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The Green Function of a Class of Degenerate Operator with Pole in the Elliptic Region

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Abstract: In this paper Green solution of a class of degenerate operator with pole in the elliptic region is obtained by the method of power series. The solution is expressed by hypergeometric function.

Keywords: degenerate operator; Green solution; hypergeometric function

1 Introduction

Consider the operator

$$G = y^l \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{1}$$

$l > 0$ in $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2, y > 0\}$. A distribution $K_{\xi,\eta}(x, y) \in \mathcal{D}'(\mathbb{R}^2_+)$ is said to be a *Green solution* of G relative to a pole $(\xi, \eta) \in \mathbb{R}^2_+$ if $K_{\xi,\eta}(x, 0) = 0$ and

$$G_{x,y} K_{\xi,\eta}(x, y) = \delta(x - \xi, y - \eta),$$

where $\delta(x - \xi, y - \eta)$ is the Dirac distribution at (ξ, η).

Green solutions are useful in studying existence, uniqueness, regularity, and other properties of partial differential equations. In this paper, we only concern the case that the pole (ξ, η) is in the elliptic region, that is, the upper half plane. In view of the invariance of G under the translations along the $x-$axis, without loss of generality, it suffices to consider the case $(\xi, \eta) = (0, b)$ with $b > 0$.

Barros-Neto and Gelfand ([4], [5], [6]) studied the reduced Tricomi operator $\mathcal{T}$ ($l = 1$ of G) in characteristic coordinates and found solutions in homogeneous form, then they obtained the fundamental solutions of $\mathcal{T}$ in original coordinates. More precisely, they obtained:

Theorem 1 *[4]. Assume $b = 0$. Then*

$$F_+(x, y) = \begin{cases} -\dfrac{1}{2^{1/3}3^{1/2}} F(\dfrac{1}{6}, \dfrac{1}{6}, 1; 1)(9x^2 + 4y^3)^{-1/6} & \text{in} \quad D_+, \\ 0 & \text{elsewhere,} \end{cases}$$

where

$$D_+ = \{(x, y) \in \mathbb{R}^2 : 9x^2 + 4y^3 > 0\},$$

and

$$F_-(x, y) = \begin{cases} \dfrac{1}{2^{1/3}} F(\dfrac{1}{6}, \dfrac{1}{6}, 1; 1)|9x^2 + 4y^3|^{-1/6} & \text{in} \quad D_-, \\ 0 & \text{elsewhere,} \end{cases}$$

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where

$$D_- = \{(x,y) \in \mathbb{R}^2 : 9x^2 + 4y^3 < 0\}$$

are two distinct fundamental solutions of the Tricomi operator $\mathcal{T}$ relative to the origin.

Assume $b > 0$ and $a = 2b^{3/2}/3$. Define

$$u(x,y) = \begin{cases} 9(x^2+a^2) + 4y^3 + 12ay^{3/2} & \text{if y} \geq 0, \\ 9(x^2+a^2) + 4y^3 - i12a(-y)^{3/2} & \text{if y} < 0, \end{cases}$$

and

$$v(x,y) = \begin{cases} 9(x^2+a^2) + 4y^3 - 12ay^{3/2} & \text{if y} \geq 0, \\ 9(x^2+a^2) + 4y^3 + i12a(-y)^{3/2} & \text{if y} < 0. \end{cases}$$

Theorem 2 *[6]. Assume $b > 0$. Then*

$$\mathcal{F}(x,y;0,b) = -\frac{1}{2^{1/3}}(-v)^{-1/6}F(\frac{1}{6},\frac{1}{6},1;\frac{u}{v}),$$

is a fundamental solution of the Tricomi operator $\mathcal{T}$ with the pole at $(0,b)$, and

$$\lim_{b\to 0} \mathrm{Re}\mathcal{F}(\mathrm{x},\mathrm{y};0,\mathrm{b}) = \frac{3}{2}\mathrm{F}_+ - \frac{1}{2}\mathrm{F}_-, \tag{2}$$

$$\lim_{b\to 0} \mathrm{Im}\mathcal{F}(\mathrm{x},\mathrm{y};0,\mathrm{b}) = -\frac{\sqrt{3}}{2}\mathrm{F}_+ + \frac{\sqrt{3}}{2}\mathrm{F}_-, \tag{3}$$

where $\mathrm{Re}\mathcal{F}$ and $\mathrm{Im}\mathcal{F}$ represent the real and imaginary parts of $\mathcal{F}$.

A natural generalized Tricomi operator is

$$\mathcal{T}_n = y\Delta_x + \frac{\partial^2}{\partial y^2},$$

where $x = (x_1, x_2, \ldots, x_n)$, $n \geq 1$. Clearly $\mathcal{T}_1 = \mathcal{T}$. In [1] and [2], Barros-Neto and Cardoso obtained the explicit formula for the fundamental solution of the generalized Tricomi operator with pole at the origin or hyperbolic region $\mathbb{R}^{n+1}_- = \{(\xi,\eta) \in \mathbb{R}^{n+1}, \eta < 0\}$, respectively.

If $y > 0$, the change of variables

$$x = x, \quad s = \frac{2}{3}y^{3/2} \tag{4}$$

transform $\mathcal{T}_n$ into the operator

$$2(\frac{3s}{2})^{2/3}Q_{1/6} \tag{5}$$

with

$$Q_\alpha = \frac{1}{2}\Delta_{x,s} + \frac{\alpha}{s}\frac{\partial}{\partial s}. \tag{6}$$

The Jacobian of the transform is

$$\frac{\partial(x,y)}{\partial(x,s)} = (\frac{2}{3})^{1/3}s^{-1/3}. \tag{7}$$

Similarly, if $y < 0$, the change of variables

$$x = x, \quad t = -\frac{2}{3}(-y)^{3/2} \tag{8}$$

transform $\mathcal{T}_n$ into the operator

$$2(\frac{-3t}{2})^{2/3}R_{1/6} \tag{9}$$

with

$$R_\alpha = \frac{1}{2}\Box_{x,t} + \frac{\alpha}{t}\frac{\partial}{\partial t}, \tag{10}$$

where $\Box_{x,t} = \frac{\partial^2}{\partial t^2} - \Delta_x$. The Jacobian of the transform is

$$\frac{\partial(x,y)}{\partial(x,t)} = (\frac{2}{3})^{1/3}(-t)^{-1/3}. \tag{11}$$

In the case $n \geq 2$, Barros-Neto and Cardoso [3] constructed the explicit formula for the fundamental solution of the generalized Tricomi operator with pole at the elliptic region $\mathbb{R}^{n+1}_+ = \{(\xi,\eta) \in \mathbb{R}^{n+1}, \eta > 0\}$. They expressed the fundamental solution of Q_α in series, then determined the coefficients of the series and got the fundamental solution of the original generalized Tricomi operator $\mathcal{T}_n$.

Take the change of variables

$$x = x, \quad s = \frac{2}{l+2}y^{\frac{l+2}{2}} \tag{12}$$

transform G into the operator

$$2(\frac{2}{l+2}s)^{\frac{2l}{l+2}}Q_\alpha \tag{13}$$

with

$$Q_\alpha = \frac{1}{2}\Delta_{x,s} + \frac{\alpha}{s}\frac{\partial}{\partial s}, \tag{14}$$

$$\alpha = \frac{l}{2(l+2)}. \tag{15}$$

The Jacobian of the transform is

$$\frac{\partial(x,y)}{\partial(x,s)} = (\frac{2}{l+2})^{\frac{l}{l+2}}s^{\frac{-l}{l+2}}. \tag{16}$$

The main result of this paper is:

Theorem 3 . *Assume*

$$y_0 > 0,\ s_0 = \frac{2}{l+2}y_0^{\frac{l+2}{2}},\ y > 0,\ s = \frac{2}{l+2}y^{\frac{l+2}{2}},$$
$$r^2 = (x-x_0)^2 + (s-s_0)^2.$$

Then $K_{x_0,y_0}(x,y)$ defined by

$$-\frac{2^{2-2\alpha}}{\pi}(\frac{2}{l+2})^{6\alpha}\frac{\Gamma^2(1-\alpha)}{\Gamma(2-2\alpha)}(s_0 s)^{-\alpha}(\frac{4s_0 s}{r^2})^{1-\alpha}F(1-\alpha,1-\alpha,2-2\alpha;-\frac{4s_0 s}{r^2}) \tag{17}$$

is the Green function of the operator G with pole at $(0, y_0)$.

The proof of Theorem 3 will be given in Section 3.

Fundamental solution is studied and used in linear and nonlinear mixed equations. Among others, S. X. Chen [8] studied the fundamental solution of Keldysh equation and Z. P. Ruan, I. Witt, H. C. Yin [17], [18] used the properties of fundamental solutions to semilinear Tricomi equations.

2 Solution to Q_α in the upper half plane

Assume

$$y_0 > 0,\ s_0 = \frac{2}{l+2} y_0^{\frac{l+2}{2}}.$$

Let $\mathcal{F}_\alpha$ be the fundamental solution of Q_α (see (14)) in the upper plane with pole at $(0, y_0)$, that is

$$Q_\alpha \mathcal{F}_\alpha = \delta(x, s - s_0) \quad \text{in } \mathbb{R}^2_+$$

Following the line of Garabedian [13] and Barros-Neto and Cardoso [3] which based on the homogeneity of Q_α, we may assume that $\mathcal{F}_\alpha$ is of the form

$$\mathcal{F}_\alpha(x, s; 0, s_0) = \log\left(\frac{r^2}{4s_0 s}\right) \cdot V + W, \tag{18}$$

where $r^2 = x^2 + (s - s_0)^2$, V and W are following series

$$V(x, s; 0, s_0) = \sum_{j=0}^{\infty} c_j s_0^{\alpha-j} s^{-\alpha-j} r^{2j}, \tag{19}$$

$$W(x, s; 0, s_0) = \sum_{j=0}^{\infty} w_j s_0^{\alpha-j} s^{-\alpha-j} r^{2j}, \tag{20}$$

where coefficients c_j are w_j are to be determined such that $\mathcal{F}_\alpha$ is a fundamental solution to Q_α. Explicit calculation implies

$$Q_\alpha\left(\log\left(\frac{r^2}{4s_0 s}\right) \cdot V\right) = 2\pi c_0 \delta(x, s - s_0) + \mathcal{L}(V) + \log\left(\frac{r^2}{4s_0 s}\right) Q_\alpha(V), \tag{21}$$

where $\mathcal{L}$ is a differential operator of first order

$$\mathcal{L} = \frac{2}{r^2}\left[x\frac{\partial}{\partial x} + (s - s_0)\frac{\partial}{\partial s} + \frac{\alpha(s - s_0)}{s}\right] - \frac{1}{s^2}\left(\alpha - \frac{1}{2}\right) - \frac{1}{s}\frac{\partial}{\partial s}.$$

Here we apply the well known result that the Laplace operator of two dimension

$$\Delta_{x,s} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

has the logarithmic singularity,

$$\Delta_{x,s}\left(\frac{1}{2\pi}\log r\right) = \delta(x, s - s_0).$$

Taking (21) into (18) yields

$$Q_\alpha \mathcal{F}_\alpha = 2\pi c_0 \delta(x, s - s_0) + Q_\alpha(W) + \mathcal{L}(V) + \log\left(\frac{r^2}{4s_0 s}\right) Q_\alpha(V). \tag{22}$$

To make the right hand side of the last equation equal to $\delta(x, s - s_0)$, we take

$$c_0 = \frac{1}{2\pi} \tag{23}$$

and

$$Q_\alpha(V) = 0,$$

$$Q_\alpha(W) + \mathcal{L}(V) = 0.$$

We first calculate $Q_\alpha(V)$. Recall

$$Q_\alpha = \frac{1}{2}\Delta_{x,s} + \frac{\alpha}{s}\frac{\partial}{\partial s},$$

we see for $j = 0$,

$$Q_\alpha(s_0^\alpha s^{-\alpha}) = \frac{\alpha(1-\alpha)}{2} s_0^\alpha s^{-\alpha-2}.$$

For $j \geq 1$, from

$$\triangle_{x,s}(r^{2j}) = 4j^2 r^{2j-2}, \qquad \frac{\partial}{\partial s}(r^{2j}) = 2j(s - s_0)r^{2j-2},$$

one gets

$$\begin{aligned} Q_\alpha(s_0^{\alpha-j} s^{-\alpha-j} r^{2j}) &= \frac{(\alpha+j)(1-\alpha+j)}{2} s_0^{\alpha-j} s^{-\alpha-j-2} r^{2j} \\ &\quad + 2j^2 s_0^{\alpha-j+1} s^{-\alpha-j-1} r^{2j-2}. \end{aligned}$$

Then

$$Q_\alpha(V) = \sum_{j=0}^{\infty} \left\{ c_j \frac{(\alpha+j)(1-\alpha+j)}{2} + c_{j+1} 2(j+1)^2 \right\} s_0^{\alpha-j} s^{-\alpha-j-2} r^{2j}.$$

To make $Q_\alpha(V) = 0$, we ask the sums in the bracket to be zero, then the induction yields

$$c_j = (-\frac{1}{4})^j \frac{(\alpha)_j (1-\alpha)_j}{j! j!} c_0. \tag{24}$$

Next we determine the coefficients of W from the equation

$$Q_\alpha(W) + \mathcal{L}(V) = 0. \tag{25}$$

Similar to the calculation of $Q_\alpha(V)$, one has

$$Q_\alpha(W) = \sum_{j=0}^{\infty} \left\{ w_j \frac{(\alpha+j)(1-\alpha+j)}{2} + w_{j+1} 2(j+1)^2 \right\} s_0^{\alpha-j} s^{-\alpha-j-2} r^{2j}.$$

On the other hand, one find

$$\mathcal{L}(V) = \sum_{j=0}^{\infty} \left\{ 4c_{j+1}(j+1) + c_j(j+1/2) \right\} s_0^{\alpha-j} s^{-\alpha-j-2} r^{2j}.$$

For convenience of calculation, let

$$w_j = -h_j c_j,$$

where h_j to be determined later. Taking above two equation into (25) and comparing the coefficients leads to

$$\begin{aligned} & h_j c_j \frac{(\alpha+j)(1-\alpha+j)}{2} + 2h_{j+1} c_{j+1} (j+1)^2 \\ & \quad = -4c_{j+1}(j+1) - c_j(j+1/2), \quad \forall j. \end{aligned} \tag{26}$$

Now (24) implies

$$c_{j+1} = -\frac{1}{4} \frac{(\alpha+j)(1-\alpha+j)}{(j+1)^2} c_j.$$

Taking it into (26) yields

$$h_{j+1} - h_j = \frac{2}{j+1} - \frac{2j+1}{(\alpha+j)(1-\alpha+j)}. \tag{27}$$

Define $\Psi(z) = \Gamma'(z)/\Gamma(z)$, where Γ is Gauss function. Then $\Psi(z)$ satisfies

$$\Psi(1+z) = \Psi(z) + \frac{1}{z}.$$

Applying the analytic continuation formula of hypergeometric series of $F(a,b,c;z)$ in the case $b = a$ (see Appendix A), if we choose

$$h_j = 2\Psi(1+j) - \Psi(1-\alpha+j) - \Psi(1-\alpha-j)$$

then h_j satisfies (27), and (25) is valid. So $\mathcal{F}_\alpha$ be the fundamental solution of Q_α.

Theorem 4 *Assume $\mathcal{F}_\alpha$ is the fundamental solution of Q_α, then it can be expressed by hypergeometric function as*

$$-\frac{2^{1-2\alpha}}{\pi}\frac{\Gamma^2(1-\alpha)}{\Gamma(2-2\alpha)} s_0 s^{-2+2\alpha} r^{-2+2\alpha} F(1-\alpha, 1-\alpha, 2-2\alpha; \frac{-4s_0 s}{r^2}). \tag{28}$$

Proof. By assumption on $0 < \alpha < \frac{1}{2}$ and the hypergeometric function is well defined. The expressions (19) and (20) of V and W and the identity $w_j = -h_j c_j$ imply in $\mathbb{R}^2_+$

$$\mathcal{F}_\alpha(x,s;x_0,s_0) = \left(\frac{s_0}{s}\right)^\alpha \sum_{j=0}^{\infty} c_j \left(\frac{r^2}{s_0 s}\right)^j \left[\log\left(\frac{r^2}{4s_0 s}\right) - h_j\right].$$

Replacing the coefficients c_j by (23) and (24) yields

$$\mathcal{F}_\alpha = -\frac{1}{2\pi}\left(\frac{s_0}{s}\right)^\alpha \sum_{j=0}^{\infty} \frac{(\alpha)_j (1-\alpha)_j}{j!j!} \left(\frac{-4s_0 s}{r^2}\right)^{-j} \left[\log\left(\frac{4s_0 s}{r^2}\right) + h_j\right].$$

The analytic continuation formula of Appendix A gives

$$\mathcal{F}_\alpha = -\frac{1}{2\pi}\left(\frac{s_0}{s}\right)^\alpha \cdot \frac{\Gamma^2(1-\alpha)}{\Gamma(2-2\alpha)} \left(\frac{4s_0 s}{r^2}\right)^{1-\alpha} F(1-\alpha, 1-\alpha, 2-2\alpha; \frac{-4s_0 s}{r^2}),$$

which leads to the desired expression

$$\mathcal{F}_\alpha(x,s;0,s_0) = -\frac{2^{1-2\alpha}}{\pi}\frac{\Gamma^2(1-\alpha)}{\Gamma(2-2\alpha)} s_0 s^{1-2\alpha} r^{-2+2\alpha} F\left(1-\alpha, 1-\alpha, 2-2\alpha; \frac{-4s_0 s}{r^2}\right).$$

■

3 Green function to the degenerate operator

Proof of Theorem 3.

Let $\mathcal{F}_\alpha$ be the fundamental solution of Q_α in $\mathbb{R}^2_+$ expressed in (28).

Define the function

$$K_{x_0,y_0}(x,y) = 2(\frac{2}{l+2})^{6\alpha} s_0^{-2\alpha} \mathcal{F}_\alpha(x,s;x_0,s_0). \tag{29}$$

Then $K_{x_0,y_0}(x,0) = 0$. Next we prove that $K_{x_0,y_0}(x,y)$ is the fundamental solution of the operator G.

Let $\phi(x,y) \in C_0^\infty(\mathbb{R}^2_+)$ is a test function supported in the upper half plane $\mathbb{R}^2_+$, we show

$$\langle K_{x_0,y_0}, G\phi\rangle = \phi(x_0,y_0). \tag{30}$$

By (29), (13), and (16) we have

$$\langle K_{x_0,y_0}, G\phi\rangle = \langle (s^{2\alpha} s_0^{-2\alpha}\mathcal{F}_\alpha, Q_\alpha\psi\rangle, \tag{31}$$

where ψ is the form of ϕ in the variables (x, s). The formal conjugate of Q_α is

$$^tQ_\alpha = \frac{1}{2}\triangle_{x,s} - \frac{\alpha}{s}\frac{\partial}{\partial s} + \frac{\alpha}{s^2}.$$

A simple calculation yields

$$^tQ_\alpha\left(s^{2\alpha}s_0^{-2\alpha}\mathcal{F}_\alpha\right) = s^{2\alpha}s_0^{-2\alpha}Q_\alpha\mathcal{F}_\alpha = \delta(x_0, s_0)$$

by Theorem 4, which and (31) imply (30). Theorem 3 is proved.

Appendix A.

In this section we recall some basic properties of hypergeometric functions needed in this paper (see [12], [22]). Denote by

$$(a)_0 = 1, \qquad (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Let a, b, and c be complex numbers, $c \neq 0, -1, -2, \ldots$. The power series

$$F(a, b, c; \zeta) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}\zeta^n$$

is called a hypergeometric series. It is a solution of the hypergeometric equation

$$\zeta(1-\zeta)\frac{d^2u}{d\zeta^2} + \{c - (a+b+1)\zeta\}\frac{du}{d\zeta} - abu = 0.$$

The series is absolutely convergent for all numbers $|\zeta| < 1$.

It can be proved that the series of $F(a, b, c; \zeta)$ is absolutely convergent for $|\zeta| = 1$ if Re$(c - a - b) > 0$ and

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

In particular,

$$F(\frac{1}{6}, \frac{1}{6}, 1; 1) = \frac{\Gamma(2/3)}{\Gamma^2(5/6)}.$$

Let $|\arg(-\zeta)| < \pi$ be the domain of the complex plane $\mathbb{C}$ minus the positive real axis. Then Barnes's contour integral extends the hypergeometric series $F(a, b, c; \zeta)$ to a single-valued analytic function of ζ in the region $|\arg(-\zeta)| < \pi$, which still denoted by $F(a, b, c; \zeta)$. In particular, if $a = b$, then

$$F(a, a, c; \zeta) = (-\zeta)^{-a}[\log(-\zeta)V(\zeta) + W(\zeta)], \tag{32}$$

where

$$V(\zeta) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\sum_{j=0}^{\infty}\frac{(a)_j(1-c+a)_j}{j!j!}\zeta^{-j}, \tag{33}$$

and

$$W(\zeta) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\sum_{j=0}^{\infty}\frac{(a)_j(1-c+a)_j}{j!j!}h_j\zeta^{-j}, \tag{34}$$

here $h_j = 2\Psi(1+j) - \Psi(\alpha+j) - \Psi(\alpha-j)$, $\Psi(z) = \Gamma'(z)/\Gamma(z)$.

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