Symbolic Computation of Solutions for the General Fifth-order KdV Equation

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Abstract: We obtain exact solutions to the general fifth-order KdV equation by using the generalized projective Riccati equations method and the Cole-Hopf transformation. Some traveling wave solutions, which include periodic and soliton solutions are derived. Some particular cases are analyzed: Sawada-Kotera, Ito, Lax and Kaup-Kupershmidt equations.

Keywords: Fifth-order KdV equation; projective Riccati equation method; Lax equation; Ito equation; Sawada-Kotera equation; Kaup-Kupershmidt equation

1 Introduction

A large variety of physical, chemical, and biological phenomena are described by nonlinear evolution equations (NLEE). The analytical study of these NLEE has relevance because the knowledge of their exact solutions facilitates the verification of numerical solvers and aids in the stability analysis of solutions as well as helps to a better understanding of solutions and helps us to understand the phenomena they describe.

The generalized fifth-order KdV equation (KdV5) reads

\[ u_t + \omega u_{xxxxx} + \alpha uu_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x = 0, \]  

where $\alpha$, $\beta$, $\gamma$ and $\omega$ are arbitrary real parameters. The KdV5 is an important mathematical model with wide applications in quantum mechanics and nonlinear optics. Typical examples are widely used in various fields such as solid state physics, plasma physics, fluid physics and quantum field theory [1–10]. Recently, the exp–function method [11] was used to find exact solutions to general KdV equation of fifth order (KdV5). On the other hand, exact solutions to KdV5 equation have been obtained using other several computational methods such as tanh method [10], generalized tanh method and extended tanh method [12–17], variational iteration method, tanh-coth method [18]. These last methods are based in the use of solutions of special Riccati equation [19], which leads to special type of solutions. The main purpose of this paper is to obtain solutions to fifth-order KdV equation by both the Projective Riccati equations method and the Cole-Hopf transformation. The first method uses the solutions of a special system of two equations, which is called projective system of Riccati equations [19] to construct a special ansatz, which was initially introduced by Conte et. al. [20]. From this ansatz we obtain solutions to the KdV5 equation. This method has been used in a satisfactory way to solve several NLEE [20–25]. Moreover, this method has been used to obtain exact solutions for two especial KdV5 equations: Kaup-Kupershmidt and Ito equations [25], however, the principal difference of our work in this paper consists on the fact that we analyzed the KdV5 equation from a more general point of view, and therefore, the solutions obtained in [25] can be derived as a particular case. It is clear that some of the solutions obtained for the projective system that we mentioned early, can be transformed to some of solutions derived using the first mentioned methods, but other solutions are new. Other important methods used to handle NLEE which are called direct, are based on Blackund transformation, Hirota bilinear form, a Lax pair. and inverse scattering transform. The last one was introduced by Ablowitz et al. [26]; however,
they are quite difficult to use and then the other methods we mentioned at the beginning are very useful in the study of NLEE.

Some important particular cases of Eq. (1) are:

- Kaup-Kupershmidt equation (KK equation) [5–9]
  \[ u_t + u_{xxxxx} + 10u u_{xxx} + 25u_x u_{xx} + 20u^2 u_x = 0. \] (2)

- Sawada-Kotera equation (SK equation) [2, 27]
  \[ u_t + u_{xxxxx} + 5u u_{xxx} + 5u_x u_{xx} + 5u^2 u_x = 0. \] (3)

- Caudrey-Dodd-Gibbon equation
  \[ u_t + u_{xxxxx} + 30u u_{xxx} + 30u_x u_{xx} + 180u^2 u_x = 0. \] (4)

- Lax equation [1]
  \[ u_t + u_{xxxxx} + 10u u_{xxx} + 20u_x u_{xx} + 30u^2 u_x = 0. \] (5)

- Ito equation [3, 4]
  \[ u_t + u_{xxxxx} + 3u u_{xxx} + 6u_x u_{xx} + 2u^2 u_x = 0. \] (6)

As the constants \( \alpha, \beta \) and \( \gamma \) change, the properties of the equation (1) drastically change. For instance, the Lax equation with \( \alpha = 10, \beta = 20, \) and \( \gamma = 30, \) and the SK equation where \( \alpha = \beta = \gamma = 5, \) are completely integrable. These two equations have \( N \)-soliton solutions and an infinite set of conserved densities. Another example is the KK equation with \( \alpha = 10, \beta = 25, \) and \( \gamma = 20, \) which is known to be integrable [7], and has bilinear representations [7, 9], but for which the explicit form of the \( N \)-soliton solutions is not known. A fourth equation in this class is the Ito equation, with \( \alpha = 3, \beta = 6, \) and \( \gamma = 2, \) which is not completely integrable, but has a limited number of special conserved densities [4]. We will find solutions of Eq. (1) for \( \omega \neq 0 \) or \( \gamma \neq 0. \)

2 The projective Riccati equation method

We search exact solutions of equation (1) in the form
\[
\begin{aligned}
u(x, t) &= v(\xi) \\
\xi &= x + \lambda t + \text{const},
\end{aligned}
\] (7)

As a result we have that the equation (1) is reduced to the nonlinear ordinary differential equation (ODE)
\[
\gamma v'(\xi)v(\xi)^2 + \alpha v^{(3)}(\xi)v(\xi) + \lambda v'(\xi) + \beta v'(\xi)v''(\xi) + \omega v^{(5)}(\xi) = 0
\] (8)

To obtain exact solution for the equation (8), we use the projective Riccati equation method [20], which may be described in the following three steps:

**Step 1.** We consider solutions of (8) in the form
\[
v(\xi) = a_0 + \sum_{j=1}^{m} \sigma(\xi)^{j-1}(a_j \sigma(\xi) + b_j \tau(\xi)),
\] (9)

where \( \sigma(\xi), \tau(\xi) \) satisfy the system
\[
\begin{cases}
\sigma'(\xi) = e \sigma(\xi) \tau(\xi) \\
\tau'^2(\xi) - \mu \sigma(\xi) + r.
\end{cases}
\] (10)

It may be verified that the first integral of this system is given by
\[
\tau^2 = -e \left[ r - 2\mu \sigma(\xi) + \frac{\mu^2 + \rho r^2 \sigma^2(\xi)}{r} \right],
\] (11)

where \( \rho = \pm 1 \) and \( e = \pm 1. \)

We consider the following solutions of the system (11).

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These are our assumptions. We assume that the following algebraic system in the variables \( \sigma \) and \( \tau \) has solutions of the form

\[
\begin{align*}
\sigma_1(\xi) &= e^{r \sec(\sqrt{\tau} \xi)}, \\
\sigma_2(\xi) &= e^{r \csc(\sqrt{\tau} \xi)}.
\end{align*}
\]

\[\begin{align*}
\sigma_3(\xi) &= e^{r \sech(\sqrt{\tau} \xi)}, \\
\sigma_4(\xi) &= e^{r \csch(\sqrt{\tau} \xi)}.
\end{align*}\]

\[\begin{align*}
\tau_1(\xi) &= \sqrt{\tau} \tan(\sqrt{\tau} \xi), \\
\tau_2(\xi) &= \sqrt{\tau} \cot(\sqrt{\tau} \xi), \\
\tau_3(\xi) &= \sqrt{\tau} \tanh(\sqrt{\tau} \xi), \\
\tau_4(\xi) &= \sqrt{\tau} \coth(\sqrt{\tau} \xi).
\end{align*}\]

**Step 2.** Substituting (9), along with (10) and (11) into (8) and collecting all terms with the same power in \( \sigma^i(\xi)\tau^j(\xi) \), we get a polynomial in the two variables \( \sigma(\xi) \) and \( \tau(\xi) \). This polynomial has the form

\[a\sigma(\xi)^m + b\sigma(\xi)^{m+3} + c\sigma(\xi)^{m+4} \tau(\xi) + d\sigma(\xi)^{3m+1} + e\sigma(\xi)^{2m+2} \tau(\xi) + \text{lower degree terms}\]

We assume that \( m \geq 1 \) to avoid trivial solutions. The degrees of the highest terms are \( m + 5 \) (the degree of the terms \( a\sigma(\xi)^m + b\sigma(\xi)^{m+3} \) and \( c\sigma(\xi)^{m+4} \tau(\xi) \)), \( 2m + 3 \) (the degree of the term \( d\sigma(\xi)^{3m+1} \) ) and \( 3m + 1 \) (the degree of the term \( e\sigma(\xi)^{2m+2} \tau(\xi) \)). There are two integer values of \( m \) for which \( 3m + 2 = 2m + 3 \) or \( 3m + 1 = m + 5 \) or \( 2m + 3 = m + 5 \). These are \( m = 1 \) and \( m = 2 \). We are going to find solutions for \( m = 1 \) (the case \( m = 2 \) is not considered here). When \( m = 1 \) solutions have the form

\[v(\xi) = a_0 + a_1\sigma(\xi) + b_1\tau(\xi)\]

and equating in (16) the coefficients of every power of \( \sigma(\xi) \) and of every term of the form \( \sigma^i(\xi)\tau^j(\xi) \) to zero, we obtain the following algebraic system in the variables \( a_0, a_1, b_1, \ldots \):

1. \( e^5(2e^2 - 1)\gamma \mu (\mu^2 + \rho)a_1 = 0. \)
2. \( e^7\gamma (\mu^2 + \rho)^2a_1 = 0. \)
3. \( e^6(2e^2 - 1)\gamma \mu (\mu^2 + \rho)^2b_1 = 0. \)
4. \( e^8\gamma (\mu^2 + \rho)^3b_1 = 0. \)
5. \( (e - 1)(e + 1)r \alpha a_0 b_1^2 = 0. \)
6. \(-120r^2 \gamma a_1 e^7 + 180r^2 \gamma a_1 e^5 - 6r \beta a_1 e^4 - 61r^2 \gamma a_1 e^3 + 4a_\mu a_0 b_1^2 e^2 - 3r \alpha a_1 b_1^2 e^2 + 5r \beta a_1 e^2 + a a_\lambda e + \lambda a_1 e - 2a_\mu a_0 b_1^2 + 2r \alpha a_1 b_1^2 = 0. \)
7. \[240 r^2 \gamma_{\mu a} e^7 - 300 r^2 \gamma_{\mu a} e^5 + 6 r \beta_{\mu a} e^4 + 75 r^2 \gamma_{\mu a} e^3 - \alpha^2 a_0 b_1 e^2 - \alpha a_0 b_1 e^2 + 3 r \alpha \mu e_2^2 - 3 r \beta_{\mu a} e^2 + \rho a_0 a_1 e^2 = 0.\]

8. \[a_1(720 r \gamma_{\mu e}^2 + 240 r \gamma_{\rho e} + 660 r \gamma_{\mu} e^2 - 180 r \gamma_{\rho e}^4 + 63 r \beta_{\rho e} e^3 + 6 r \gamma_e^3 e + 90 r \gamma_{\mu} e^2 + 3 a_{\mu e}^2 b_1 e + 3 a_{\rho e} b_1 e - r a_0 a_1^2) = 0.\]

9. \[(e - 1)(e + 1) r b_1 (120 r^2 \gamma_e^2 - 120 r^2 \gamma_e^4 + 6 r \beta e^3 + 16 r^2 \gamma e^2 + r a_0 b_1 e - 2 r \beta e - \alpha a_0^2 - \lambda) = 0.\]

10. \[-b_1(720 r \gamma_{\mu e}^2 + 1320 r \gamma_{\mu e}^2 + 24 r \beta_{\mu e}^3 + 662 r^2 \gamma_{\mu e}^4 + 4 r a_{\mu e} b_1 e^3 - 28 r \beta_{\mu e}^3 - 2 a_{\mu a_0} e_2 - 2 \lambda a_{\mu e}^2 e - 2 \gamma_{\mu e}^2 + 4 r a_{\mu a_1} e^2 - 3 r a_{\mu b_1 e} + 5 r \beta_{\mu e} + a_{\mu a_0}^2 + \lambda a_{\mu} - 2 r a_{\mu a_0 a_1}) = 0.\]

11. \[b_1(1800 r^2 \gamma_{\mu e}^8 + 360 r^2 \gamma_{\rho e}^8 - 2880 r^2 \gamma_{\mu} e^6 - 480 r^2 \gamma_{\rho e}^6 + 36 r^2 \beta_{\mu e}^2 e^5 + 12 r \beta_{\rho e}^5 + 1186 r^2 \gamma_{\mu} e^4 + 136 r^2 \gamma_{\rho e}^4 - 32 r \beta_{\mu e} e^3 + 6 r a_{\mu e} b_1 e^3 + 2 r a_{\rho e} b_1 e^3 - 8 r \beta_{\rho e} e^3 - 72 r^2 \gamma_{\mu} e^2 - \lambda r e e^2 - \alpha^2 a_0^2 e^2 - \alpha a_0 b_1^2 e - 3 r a_{\rho a_0} e_2^2 - 3 r a_{\mu} e_2^2 - \lambda r e^2 + 8 r a_{\rho a_0 a_1} e^2 + 3 r a_{\mu b_1 e} - 3 r a_{\mu b_1} e_2 - 3 r a_{\rho b_1} e + r^2 a_1^2 - 2 r a_{\mu a_0 a_1}) = 0.\]

12. \[b_1(2400 r^2 \gamma_{\mu e}^8 + 1440 r^2 \gamma_{\rho e}^8 - 3120 r^2 \gamma_{\mu} e^6 - 1680 r^2 \gamma_{\rho e}^6 + 24 r^2 \gamma_{\mu} e^4 + 24 r^2 \gamma_{\rho e}^4 - 930 r^2 \gamma_{\mu} e^4 + 90 r^2 \gamma_{\rho e}^4 - 12 r^2 \gamma_{\mu} e^3 + 4 a_{\mu e} b_1 e^3 + 4 a_{\rho e} b_1 e^3 - 12 r^2 \gamma_{\rho e} e^3 - 30 r^2 \gamma_{\mu} e^2 - 6 r a_{\mu a_0} e^2 + 4 a_{\mu a_1} b_1 e^2 + 4 a_{\rho a_0} a_1 e^2 - a_{\mu b_1} e^2 - a_{\rho b_1} e + r a_{\mu a_1}^2 = 0.\]

13. \[(\mu^2 + \rho) b_1(1800 r^2 \gamma_{\mu e}^6 + 360 r^2 \gamma_{\rho e}^6 - 1680 r^2 \gamma_{\mu} e^4 - 240 r^2 \gamma_{\rho e}^4 + 6 r^2 \mu e^6 + 6 r^2 \beta e^3 + 270 r^2 \gamma_{\mu} e^2 + a_{\mu b_1} e^2 + a_{\rho b_1} e + r a_{\mu a_1}^2 = 0.\]

**Step 3.** (This is the more difficult step) Solving the previous system for \(r, \mu, a_0, a_1, b_1\), we get \(b_1 = 0\), so solutions have the form \(u = a_0 + a_1 \sigma(x + \lambda t)\). We get the following solutions to (1), where \(\xi_0\) represents an arbitrary constant and

\[
A = 2 \alpha + \beta + \sqrt{(2 \alpha + \beta)^2 - 40 \gamma_{\omega}}, \quad B = \frac{12 \gamma_{\omega} - A \beta}{8 \gamma} \quad \text{and} \quad C = \frac{(3 A - 10 \beta) \gamma}{2 A}.
\]

**A)** \(e = 1\) and \(\rho = -1\):

- \(a_0 = \frac{10 \omega}{A}, a_1 = -\frac{60 \omega}{A}, \lambda = C r^2, \mu = 1:\)
  \[u_1(x, t) = \frac{10 \omega}{A} \left(1 - 3 \sec^2 \left(\frac{1}{2} \sqrt{r} \left(x + C r^2 t + \xi_0\right)\right)\right).\]  
  \[u_2(x, t) = -\frac{10 \omega}{A} \left(5 - \frac{6}{1 + \csc \left(\sqrt{r} \left(x + C r^2 t + \xi_0\right)\right)}\right).\]

- \(a_0 = \frac{A}{4 \gamma}, a_1 = -\frac{3 A}{2 \gamma}, \lambda = B r^2, \mu = 1:\)
  \[u_3(x, t) = \frac{A}{4 \gamma} \left(1 - 3 \sec^2 \left(\frac{1}{2} \sqrt{r} \left(x + B r^2 t + \xi_0\right)\right)\right).\]
  \[u_4(x, t) = -\frac{A}{4 \gamma} \left(5 - \frac{6}{1 + \csc \left(\sqrt{r} \left(x + B r^2 t + \xi_0\right)\right)}\right).\]

- \(a_0 = \frac{10 \omega}{A}, a_1 = \frac{60 \omega}{A}, \lambda = C r^2, \mu = -1:\)
  \[u_5(x, t) = \frac{10 \omega}{A} \left(1 - 3 \sec^2 \left(\frac{1}{2} \sqrt{r} \left(x + C r^2 t + \xi_0\right)\right)\right).\]
  \[u_6(x, t) = -\frac{10 \omega}{A} \left(5 - \frac{6}{1 - \csc \left(\sqrt{r} \left(x + C r^2 t + \xi_0\right)\right)}\right).\]

- \(a_0 = \frac{A}{4 \gamma}, a_1 = \frac{3 A}{2 \gamma}, \lambda = B r^2, \mu = -1:\)
  \[u_7(x, t) = \frac{A}{4 \gamma} \left(1 - 3 \sec^2 \left(\frac{1}{2} \sqrt{r} \left(x + B r^2 t + \xi_0\right)\right)\right).\]
  \[u_8(x, t) = -\frac{A}{4 \gamma} \left(5 - \frac{6}{1 - \csc \left(\sqrt{r} \left(x + B r^2 t + \xi_0\right)\right)}\right).\]
B) $e = -1$ and $\rho = -1$:

- $a_0 = -\frac{10r\omega}{A}, a_1 = \frac{60\omega}{A}, \lambda = Cr^2, \mu = 1$:
  \[ u_9(x, t) = \frac{-10r\omega}{A} \left( 1 - 3 \text{sech}^2 \left( \frac{1}{2} \sqrt{r} \left( x + Cr^2 t + \xi_0 \right) \right) \right). \quad (27) \]

- $a_0 = -\frac{Ar}{4\gamma}, a_1 = \frac{3A}{2\gamma}, \lambda = Br^2, \mu = 1$:
  \[ u_{10}(x, t) = \frac{-Ar}{4\gamma} \left( 1 - 3 \text{sech}^2 \left( \frac{1}{2} \sqrt{r} \left( x + Br^2 t + \xi_0 \right) \right) \right). \quad (28) \]

- $a_0 = -\frac{10r\omega}{A}, a_1 = -\frac{60\omega}{A}, \lambda = Cr^2, \mu = -1$:
  \[ u_{11}(x, t) = \frac{-10r\omega}{A} \left( 1 + 3 \text{csch}^2 \left( \frac{1}{2} \sqrt{r} \left( x + Cr^2 t + \xi_0 \right) \right) \right). \quad (29) \]

- $a_0 = -\frac{Ar}{4\gamma}, a_1 = -\frac{3A}{2\gamma}, \lambda = Br^2, \mu = -1$:
  \[ u_{12}(x, t) = \frac{-Ar}{4\gamma} \left( 1 + 3 \text{csch}^2 \left( \frac{1}{2} \sqrt{r} \left( x + Br^2 t + \xi_0 \right) \right) \right). \quad (30) \]

C) $e = -1$ and $\rho = 1$:

- $a_0 = -\frac{10r\omega}{A}, a_1 = -\frac{60\omega i}{A}, \lambda = Cr^2, \mu = -i$:
  \[ u_{13}(x, t) = \frac{10r\omega}{A} \left( 5 - \frac{6}{1 - \csc \left( \sqrt{-r} \left( x + Cr^2 t + \xi_0 \right) \right)} \right). \quad (31) \]

- $a_0 = -\frac{Ar}{4\gamma}, a_1 = -\frac{3A}{2\gamma}, \lambda = Br^2, \mu = -i$:
  \[ u_{14}(x, t) = \frac{Ar}{4\gamma} \left( 5 - \frac{6}{1 - \csc \left( \sqrt{-r} \left( x + Br^2 t + \xi_0 \right) \right)} \right). \quad (32) \]

- $a_0 = -\frac{10r\omega}{A}, a_1 = \frac{60\omega i}{A}, \lambda = Cr^2, \mu = i$:
  \[ u_{15}(x, t) = \frac{10r\omega}{A} \left( 5 - \frac{6}{1 + \csc \left( \sqrt{-r} \left( x + Cr^2 t + \xi_0 \right) \right)} \right). \quad (33) \]

- $a_0 = -\frac{Ar}{4\gamma}, a_1 = \frac{3A}{2\gamma}, \lambda = Br^2, \mu = i$:
  \[ u_{16}(x, t) = \frac{Ar}{4\gamma} \left( 5 - \frac{6}{1 + \csc \left( \sqrt{-r} \left( x + Br^2 t + \xi_0 \right) \right)} \right). \quad (34) \]
3 Soliton solutions by the Cole-Hopf transformation

We seek two-soliton solutions to equation (1) in the Cole-Hopf form

\[
\left\{
\begin{aligned}
    u = u(x, t) &= A \partial_{xx} \log(1 + a \exp(kx - wt)) + B, \\
    A, B, a, k, w &= \text{const. (35)}
\end{aligned}
\right.
\]

Inserting ansatz (35) into equation (1) we get a polynomial equation in the variable \(\zeta = \exp(kx - wt)\). Equating to zero the coefficients of \(\zeta^j (j = 0, 1, 2, \ldots)\) gives an algebraic system in the variables \(A, B, a, k, w\). This system reads

\[
\left\{
\begin{aligned}
    B^2 k\gamma + Bk^3\alpha + k^5\omega - w &= 0, \\
    2ABk^3\gamma + Ak^5\alpha + Ak^5\beta + 3B^2 k\gamma - 9Bk^3\alpha - 57k^5\omega - 3w &= 0, \\
    A^2 k^5\gamma + 2ABk^3\gamma - 11Ak^5\alpha - 5Ak^5\beta + 2B^2 k\gamma - 10Bk^3\alpha + 302k^5\omega - 2w &= 0.
\end{aligned}
\right.
\]

Solutions to system (36) are:

\[
\left\{
\begin{aligned}
    A &= \frac{3}{\gamma} \left(2\alpha + \beta \pm \sqrt{(2\alpha + \beta)^2 - 40\gamma \omega}\right), \\
    B &= -\frac{k^2}{4\gamma} \left(2\alpha + \beta \pm \sqrt{(2\alpha + \beta)^2 - 40\gamma \omega}\right), \\
    w &= \frac{k^5}{8\gamma} \left(\beta \left(2\alpha + \beta \pm \sqrt{(2\alpha + \beta)^2 - 40\gamma \omega}\right) - 12\gamma \omega\right).
\end{aligned}
\right.
\]

Finally, from (35) and (37) we conclude that following is a solution to the general KdV5 (1):

\[
u = u_{18}(x, t) = \frac{k^2 c}{4\gamma} + \frac{3ck^2}{4\gamma} \cdot \frac{1}{1 + a \exp \xi} - \frac{3ck^2}{4\gamma} \cdot \frac{1}{(1 + a \exp \xi)^2},
\]  
\]  

where

\[
c = 2\alpha + \beta \pm \sqrt{(2\alpha + \beta)^2 - 40\gamma \omega}
\]  

and \(\xi = kx - \frac{k^5}{8\gamma} (\beta c - 12\gamma \omega) t\).

4 Analysis of the obtained results

The identities

\[
\sec^2 \theta = 1 - \tan^2 \theta, \quad \csc^2 \theta = 1 - \cot^2 \theta,
\]

\[
5 - \frac{6}{1 \pm \csc(\theta)} = 2 + 3 \cot^2 \left(\frac{\theta}{2} \pm \frac{\pi}{4}\right).
\]

we see that all solutions are expressible in terms of \(\tanh\) and \(\coth\). Comparing this with the results in [16], we conclude that the projective Riccati equations method and the extended tanh-coth methods give the same solutions. This always happens when \(\mu = \pm 1\).

On the other hand, for \(a > 0\),

\[
u_{18}(x, t) = -\frac{k^2 c}{4\gamma} + \frac{3ck^2}{4\gamma} \sech^2 \left(\frac{1}{2}(\xi + \log(a))\right),
\]  

and for \(a < 0\),

\[
u_{18}(x, t) = -\frac{k^2 c}{4\gamma} - \frac{3ck^2}{4\gamma} \csc \left(\frac{1}{2}(\xi + \log(-a))\right).
\]

We again see that results coincide within those we obtain by using the tanh-coth method.
5 Conclusions

In this paper, with the aid of a symbolic computation engine, we obtained some exact solutions to equation (1). At the same time, we may find solutions for any particular case of this equation, for example, (2)-(6). We also may say that solutions coincide with those we obtain by both the extended tanh-coth method and the Cole-Hopf transformation method. Other methods to solve nonlinear pde’s exactly may be found in [28, 38].

References


