The Exact Solitary Wave Solutions for a Family of BBM Equation

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Abstract: In this paper, the family of BBM equation with strong nonlinear dispersive $B(m,n)$ equation is introduced. By using a simple and effective algebraic method, some abundant exact solutions for this class of BBM equations are gotten and call its solutions with solitary patterns, solitary wave solution with singular point, smooth solitary wave solution, kink solution, anti-kink solution, floating solitary wave solution, peakon solution, compacton solution.

Keywords: BBM equation; solitary wave; Peakon; Compacton; strong dispersion

1 Introduction

To study the role of nonlinear dispersive in the formation of patterns in liquid drops, Rosenau and Hyman (see\cite{1}) studied the genuinely nonlinear dispersive equations $K(m,n)$ given by

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m > 0, \quad 1 < n \leq 3,$$

and introduced a class of solitary wave solutions with compact support that are solutions of a two parameter family of fully nonlinear dispersive partial differential equations such as $K(2,2)$ equation

$$u_t + (u^2)_x + (u^2)_{xxx} = 0,$$

and

$$u_t + (u^3)_x + (u^3)_{xxx} = 0.$$

Recently, Wazwaz (see\cite{2}) gave exact special solutions with solitary patterns for the nonlinear dispersive $K(m,n)$ equations:

$$u_t - (u^m)_x + (u^n)_{xxx} = 0, \quad m, n > 1.$$

In\cite{3}, Wazwaz studied the genuinely nonlinear dispersive $K(m,n)$ equations

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m, n > 1,$$

and obtained the new solitary wave special solutions with compact support of the nonlinear dispersive $K(m,n)$ equation.

Yan and Bluman (see\cite{4}), Yan (see\cite{5}) and Zhu (see\cite{6}) investigated a family of Boussinesq-like equations with fully nonlinear dispersion $B(m,n)$ equations: $u_{tt} = (u^m)_{xx} + (u^n)_{xxx}$ and $u_{tt} - (u^m)_{xx} + (u^n)_{xxx} = 0$. New families of solitons with compact support and solitary patterns solutions for Boussinesq-like $B(m,n)$ equations with fully nonlinear dispersion are developed by the Adomian decomposition method, respectively.

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Tian, Gui and Liu (see[7]) solved the well-posedness problem, the isospectral problem and the scattering problem for the Dullin-Gottwald-Holm (DGH) equation. Meanwhile, the scattering data of the scattering equation for the equation was explicitly expressed.

Tian and Yin (see[8]) introduced a fifth-order K(m,n) equation with nonlinear dispersion to obtain multi-compaction solutions by the Adomian decomposition method. And in [9], Fan and Tian proved that solitary wave of mKdV-KS equation persisted when the perturbation parameter was suitably small. Tian, Yin (see[10][11]) considered the nonlinear generalized Camassa-Holm equation and derived some new compacton and floating compacton solutions by using four direct ansatzs. Tian and Song (see[12]) considered generalized Camassa-Holm equations and generalized weakly dissipative Cammassa-Holm equations, and derived some new exact peaked solitary wave solutions. Ding and Tian (see[13]) studied the existence of global solution and got the existence of the global attractor on dissipative Cammassa-Holm equation.

BBM equation or the regularized long-wave equation:

$u_t + uu_x - u_{xxx} = 0$,

was put forward by Peregrine (see[14][15]) and Benjamin et al.(see[16]) as an alternative model to the Korteweg-de Vries equation for small-amplitude, long wavelength surface water waves.

Shang (see[17]) studied the BBM-like equations with fully nonlinear dispersion, B(m,n) equations:

$u_t + (u^m)_x - (u^n)_{xxx} = 0, \quad m, n > 1$.

The exact solitary-wave solutions with compact support and exact special solutions with solitary patterns of the equations are derived.

In this paper, we introduce and study a family of BBM equation with strong nonlinear dispersive B(m,n) equation:

$u_t + u_x + a(x^m)_x + (u^n)_{xxx} = 0$.

Some abundant exact solutions for the equations are gotten.

The rest of this paper is organized as follows: In section 2, we give many types of solitary pattern solutions of B(m,n) equations. In section 3 and section 4, we derive peakon solution and compacton solution for the family of equations.

## 2 Solitary pattern solutions of B(m,n) equations

First of all, we make the traveling wave transformation as follows:

$u(x, t) = u(\xi), \quad \xi = \lambda_1 x + \lambda_2 t$.

(2.2)

where $\lambda_i (i = 1, 2, 3)$ are parameters to be determined later. Integrating (1) and setting integration constants to zero, we have

$\lambda_2 u + \lambda_1 u + a\lambda_1 u^m + \lambda_1^2 \lambda_2 (u^n) \xi = 0$.

(2.3)

To seek solitary pattern solutions we assume that has the solution in the form

$u(x, t) = u(\xi) = A \sin h^B(\xi)$.

(2.4)

$u(x, t) = u(\xi) = A \cos h^B(\xi)$.

(2.5)

$u(x, t) = u(\xi) = A \tan h^B(\xi)$.

(2.6)

where $A, B$ are parameters to be determined later.

### 2.1 The first type of solitary pattern solution

Substituting (2.3) into (2.2), we get the following polynomial equations

$\lambda_2 A \sin h^B(\xi) + \lambda_1 \sin h^B(\xi) + a\lambda_1 A^m \sin h^m B(\xi) + \lambda_1^2 \lambda_2 A^n nB(nB - 1) \sin h^{nB-2}(\xi)$

$+ \lambda_1^2 \lambda_2 A^n n^2 B^2 \sin h^{nB}(\xi) = 0$.

(2.7)
Comparing the coefficients of similar terms we have
\[ n = 1, \quad B = \frac{2}{1 - m}, \quad A^{n-1} = \frac{2\lambda_1^2(1 + m)}{a[4\lambda_1^2 + (1 - m)^2]}, \quad \lambda_2 = -\frac{\lambda_1(1 - m)^2}{4\lambda_1^2 + (1 - m)^2}, \] (2.8)
or
\[ m = n, \quad B = \frac{2}{n - 1}, \quad A^{n-1} = \frac{4\lambda_1^2n^2 - a(n - 1)^2}{2a\lambda_1^2n(n + 1)}, \quad \lambda_2 = -\frac{a(n - 1)^2}{4\lambda_1n^2}. \] (2.9)

By combining (2.3),(2.7) and (2.8), we obtain
\[ u_1(x, t) = \left[ \frac{2\lambda_1^2(1 + m)}{a[4\lambda_1^2 + (1 - m)^2]} \right] \cdot \sin h^{\lambda_2} \left[ \lambda_1 x - \frac{\lambda_1(1 - m)^2}{4\lambda_1^2 + (1 - m)^2} t \right], \] (2.10)
or
\[ u_2(x, t) = \left[ \frac{4\lambda_1^2n^2 - a(n - 1)^2}{2a\lambda_1^2n(n + 1)} \right] \cdot \sin h^{\lambda_2} \left[ \lambda_1 x - \frac{a(n - 1)^2}{4\lambda_1n^2} t \right]. \] (2.11)

(2.9),(2.10) describe solitary wave solution with singular point and solitary pattern solutions for the equation (1) respectively. (see Fig.1 and Fig.2 )

Figure 1: Graphics of solution (10)

Figure 2: Graphics of solution (11)

2.2 The second type of solitary pattern solution

Substituting (2.4) into (2.2)
\[ \lambda_2 A \cos h^B(\xi) + \lambda_1 A \cos h^B(\xi) + a\lambda_1 A^m \cos h^{nB}(\xi) \]
\[ + \lambda_1^2 \lambda_2 A^n B^2 \cos h^{nB}(\xi) - \lambda_1^2 \lambda_2 A^n nB(nB - 1) \cos h^{nB-2}(\xi) = 0. \] (2.12)

Comparing the coefficients of similar terms we have
\[ m = n, \quad B = \frac{2}{n - 1}, \quad A^{n-1} = \frac{a(n - 1)^2 - 4\lambda_1^2n^2}{2a\lambda_1^2(n + 1)}, \quad \lambda_2 = -\frac{a(n - 1)^2}{4\lambda_1n^2}, \] (2.13)
or
\[ n = 1, \quad B = \frac{2}{1 - m}, \quad A^{m-1} = \frac{2}{a} \frac{\lambda_1^2(1 + m)}{4\lambda_1^2 + (1 - m)^2}. \] (2.14)
By combing (2.4), (2.12) and (2.13), we obtain
\[ u_3(x, t) = \left[\frac{a(n - 1)^2 - 4\lambda_1^2 n^2}{2a\lambda_1^2 n(n + 1)}\right]^{\frac{1}{n-1}} \cdot \cos h^{\frac{2}{n-1}} \left[\lambda_1 x - \frac{a(n - 1)^2}{4\lambda_1 n^2} t\right], \]  
(2.15)

or
\[ u_4(x, t) = \left[\frac{-2\lambda_1^2 (1 + m)}{a[4\lambda_1^2 + (1 - m)^2]}\right]^{\frac{1}{n-1}} \cdot \cos h^{\frac{2}{n-1}} \left[\lambda_1 x - \frac{\lambda_1 (1 - m)^2}{4\lambda_1^2 + (1 - m)^2} t\right]. \]  
(2.16)

(2.14)(2.15) describe solitary pattern solutions and smooth solitary wave solution for the equation (1) respectively. (see Fig.3)

2.3 The third type of solitary pattern solution
Substituting (2.5) into (2.2):
\[ (\lambda_2 + \lambda_1)A \tan h^B(\xi) + a\lambda_1 A^m \tan h^{mB}(\xi) + \lambda_1^2 \lambda_2 A^n B(nB - 1) \tan h^{nB-2}(\xi) \]
\[ -2\lambda_1^2 \lambda_2 A^n B^2 \tan h^{nB}(\xi) + \lambda_1^2 \lambda_2 A^n (nB + 1) B \tan h^{nB+2}(\xi) = 0. \]  
(2.17)

Comparing the coefficients of similar terms we have
\[ B = 1, \ n = 1, \ m = 3, \ A^2 = \frac{2\lambda_1^2}{a(2\lambda_1^2 - 1)}, \ \lambda_2 = \frac{\lambda_1}{2\lambda_1^2 - 1}, \]  
(2.18)

or
\[ m = n, \ nB = 1, \ B = 3, \ A^{n-1} = \frac{-a + 2\lambda_1^2}{2a\lambda_1^2}, \ \lambda_2 = \frac{a}{2\lambda_1}, \]  
(2.19)

or
\[ n = 1, \ B = 2, \ m = 2, \ A = \frac{-4\lambda_1^2}{a(4\lambda_1^2 - 1)}, \ \lambda_2 = \frac{\lambda_1}{4\lambda_1^2 - 1}, \]  
(2.20)

or
\[ B = 4, \ m = \frac{1}{2}, \ m = n, \ A^{n-1} = \frac{-4\lambda_1^2 + a}{4a\lambda_1^2}, \ \lambda_2 = \frac{a}{4\lambda_1}, \]  
(2.21)

By combing (2.5), (2.7), (2.18), (2.19) and (2.19), we obtain
\[ u_5(x, t) = \left[\frac{2\lambda_1^2}{a(2\lambda_1^2 - 1)}\right]^{\frac{1}{2}} \cdot \tan h \left[\lambda_1 x + \frac{\lambda_1}{2\lambda_1^2 - 1} t\right], \]  
(2.22)

or
\[ u_6(x, t) = \left[\frac{-a + 2\lambda_1^2}{2a\lambda_1^2}\right]^{\frac{1}{2}} \cdot \tan h^3 \left[\lambda_1 x + \frac{a}{2\lambda_1} t\right], \]  
(2.23)

\[ u_7(x, t) = -\frac{4\lambda_1^2}{a(4\lambda_1^2 - 1)} \cdot \tan h^2 \left[\lambda_1 x + \frac{\lambda_1}{4\lambda_1^2 - 1} t\right], \]  
(2.24)

or
\[ u_8(x, t) = -\frac{-4\lambda_1^2 + a}{4a\lambda_1^2} \cdot \tan h^2 \left[\lambda_1 x + \frac{a}{4\lambda_1} t\right]. \]  
(2.25)

(2.21),(2.22),(2.23) and (2.24) describe kink solution, anti-kink solution, floating solitary wave solution for the equation (1) respectively. (see Fig.4, Fig.5, Fig.6, Fig.7)

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3 Peakon solution of B(m,n) equations

First of all we make the traveling wave transformation as follows:

\[ u(x, t) = u(\xi), \; \xi = x - Dt. \]  \hspace{1cm} (3.26)

Integrating Eq.(1.1) and setting integration constants to zero, we have

\[ -Du + u + au^m - D(u^n)_{\xi\xi} = 0. \]  \hspace{1cm} (3.27)

Let \( u(\xi) = \lambda e^{-b|\xi|}, \) (3.2) reduces to be

\[ e^{-b\xi}\lambda - De^{-b\xi}\lambda + (e^{-b\xi})^m a\lambda^m + b^2 De^{-2bn\xi} n\lambda^n - b^2 De^{-bn\xi} n\lambda^n - b^2 De^{-2bn\xi} n^2 \lambda^n = 0. \]  \hspace{1cm} (3.28)

Comparing the coefficients of similar terms we have

\[ m = n, \; D = 1, \; b = \pm \sqrt{a/n}, \]

then \( u_9(x, t) = \lambda e^{\pm \sqrt{a/n}|x-t|}. \)

(3.4) describes peakon solution for the equation (1.1). (see Fig.8)
Figure 7: Graphics of solution (25)

Figure 8: Graphics of solution (29)

4 Compacton solution of B(m,n) equations

Let

\[ u(\xi) = \begin{cases} 
  A \cos^q(B\xi), & |B\xi| \leq \frac{\pi}{2}, \\
  0, & |B\xi| \geq \frac{\pi}{2}.
\end{cases} \]  

(4.29)

Substituting (4.1) into (3.2):

\[(1 - D)A \cos^q(B\xi) + aA^m \cos^{mq}(B\xi) + DA^m B^2 n^2 q^2 \cos^{nq}(B\xi) - DA^m B^2 nq(nq - 1) \cos^{nq-2}(B\xi) = 0.\]

Comparing the coefficients of similar terms we have

\[ m = n, \quad q = \frac{2}{n-1}, \quad A = \left(-\frac{a}{D-1}\right)^{\frac{1}{1-n}}, \]

where \( B \) is another arbitrary constant.

\[ u_{10}(\xi) = \begin{cases} 
  \left(-\frac{a}{D-1}\right)^{\frac{1}{1-n}} \cos^{\frac{2}{n-1}}, & |B\xi| \geq \frac{\pi}{2}, \\
  0, & |B\xi| \leq \frac{\pi}{2}.
\end{cases} \]  

(4.30)

(4.2) describes compacton solution for the equation (1.1). (see Fig.10)
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References


