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Existence of Three Solutions for a Boundary Value Problem in One-dimensional Case

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Abstract: In this note, the existence of at least three weak solutions for Dirichlet problem

$$\begin{cases} u'' + \lambda f(x,u) = 0, & x \in (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

where $\lambda > 0$ and $f : [0,1] \times R \to R$ is a continuous function, is established. The approach is based on variational methods and critical points.

Key words: Three solutions; Critical point; Multiplicity results; Dirichlet problem.
AMS subject classification: 35J20; 34A15.

1 Introduction

In this work, we study the boundary value problem

$$\begin{cases} u'' + \lambda f(x,u) = 0, & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

where $\lambda > 0$ and $f : [0,1] \times R \to R$ is a continuous function.

Precisely, we deal with the existence of an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q, such that, for each $\lambda \in \Lambda$, problem (1) admits at least three weak solutions whose norms in $W_0^{1,2}([0,1])$ are less than q.

We say that u is a weak solution to (1) if $u \in W_0^{1,2}([0,1])$ and

$$\int_0^1 u'(x)v'(x)dx - \lambda \int_0^1 f(x,u(x))v(x)dx = 0$$

for every $v \in W_0^{1,2}([0,1])$.

Multiplicity results for the problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and $f : \Omega \times R \to R$ is a Caratheodory function (see, for example, [1]) and in the case $N = 1,\ p = 2$ (see, for example, [3]) have been broadly investigated in recent years. For instance, in [1], using variational methods, the authors ensure the existence of a sequence of

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arbitrarily small positive solutions for problem (2) when the function f has a suitable oscillating behaviour at zero. Also, in [3], the author proves multiplicity results for the problem

$$\begin{cases} u'' + \lambda f(u) = 0 \\ u(0) = u(1) = 0 \end{cases} \tag{3}$$

which for each $\lambda \in [0, +\infty[$, admits at least three solutions in $W_0^{1,2}([0,1])$ when $f : R \to R$ is continuous function.

In the present paper, our approach is based on a three critical points theorem proved in [5], recalled below for the reader's convenience (Theorem 2.1), on a technical lemma (lemma 2.2) that allow us to apply it.

Theorem 2.3 which is our main result, ensures the existence of an open interval $\Lambda \subseteq [0, \infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1) admits at least three weak solutions whose norms in $W_0^{1,2}([0,1])$ are less than q.

As a consequence of Theorem 2.3, we obtain Theorem 2.4.

Theorem 2.4 ensures the existence of weak solutions for the boundary value problem (3).

The aim of the present paper is to extend the main result of [3] for problem (1).

2 Main results

First, we want to point out that our main tool is a critical points theorem, which we here recall in its equivalent formulation [2, Theorem 1.1 and Remark 1.1]:

Theorem 2.1. Let X be a separable and reflexive real Banach space; $\Phi : X \longrightarrow R$ a continuously $G\widehat{a}teaux$ differentiable and sequentially weakly lower semicontinuous functional whose $G\widehat{a}teaux$ derivative admits a continuous inverse on X^*; $\Psi : X \longrightarrow R$ a continuously $G\widehat{a}teaux$ differentiable functional whose $G\widehat{a}teaux$ derivative is compact.

Assume that

$$\lim_{||u|| \to +\infty} (\Phi(u) + \lambda \Psi(u)) = +\infty$$

for all $\lambda \in [0, +\infty[$, and that there exists $\rho \in R$ such that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda \Psi(u) + \rho\lambda) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda \Psi(u) + \rho\lambda).$$

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda \Psi'(u) = 0$$

has at least three solutions in X whose norms are less than q.

Here and in the sequel, X will denote the Sobolev space $W_0^{1,2}([0,1])$ with the norm

$$|| u || = \left(\int_0^1 |u'(x)|^2 dx \right)^{1/2},$$

and put

$$g(x,t) = \int_0^t f(x,\xi) d\xi$$

for each $(x,t) \in [0,1] \times R$.

Our main results fully depend on the following lemma:

Lemma 2.2. Assume that there exist two positive constants c and d with $c < \frac{d}{\sqrt{2}}$, such that:

(i) $g(x,t) \geq 0$ for each $(x,t) \in [0, \frac{1}{2}] \times [0, d]$,

(ii) $\frac{1}{2c^2} \max_{(x,t)\in[0,1]\times[-c,c]} g(x,t) < \frac{1}{d^2} \int_{\frac{1}{2}}^{1} g(x,d)dx.$

Then, there exist $r > 0$ and $w \in X$ such that $||w||^2 > 2r$ and

$$\max_{(x,t)\in[0,1]\times[-\sqrt{r/2},\sqrt{r/2}]} g(x,t) < 2r\frac{\int_0^1 g(x,w(x))dx}{||w||^2}.$$

Proof: We put

$$w(x) = \begin{cases} 2dx & 0 \leq x \leq \frac{1}{2}, \\ d & \frac{1}{2} \leq x < 1, \\ 0 & x = 1 \end{cases}$$

and $r = 2c^2$. It is easy to see that $w \in X$ and, in particular, one has

$$||w||^2 = 2d^2.$$

Hence, taking into account that $c < \frac{d}{\sqrt{2}}$, one has

$$2r < ||w||^2.$$

Since $0 \leq w(x) \leq d$ for each $x \in \Omega$, condition (i) ensures that

$$\int_0^{\frac{1}{2}} g(x,w(x))dx \geq 0. \tag{4}$$

Moreover, from (ii) and (4), we have

$$\max_{(x,t)\in[0,1]\times[-\sqrt{r/2},\sqrt{r/2}]} g(x,t) < (\frac{\sqrt{2}c}{d})^2 \int_{\frac{1}{2}}^{1} g(x,d)dx \leq 2r\frac{\int_0^1 g(x,w(x))dx}{||w||^2}.$$

Namely

$$\max_{(x,t)\in[0,1]\times[-\sqrt{r/2},\sqrt{r/2}]} g(x,t) < 2r\frac{\int_0^1 g(x,w(x))dx}{||w||^2}.$$

So, the proof is complete. □

Now, we state our main result:

Theorem 2.3. Assume that there exist three positive constants c, d, s with $c < \frac{d}{\sqrt{2}}$, $s < 2$ and $a \in L^1$ which $a(x) \geq 0$ 0n $[0, \frac{1}{2}]$ such that:

(i) $g(x,t) \geq 0$ for each $(x,t) \in [0, \frac{1}{2}] \times [0, d]$,

(ii) $\frac{1}{2c^2} \max_{(x,t)\in[0,1]\times[-c,c]} g(x,t) < \frac{1}{d^2} \int_{\frac{1}{2}}^{1} g(x,d)dx.$

(iii) $g(x,t) \leq a(x)(1 + |t|^s)$ almost everywhere in $[0,1]$ and for each $t \in R$.

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1) admits at least three solutions in X whose norms are less than q.

Proof: For each $u \in X$, we put

$$\Phi(u) = \frac{||u||^2}{2},$$

$$\Psi(u) = -\int_0^1 g(x, u(x))dx.$$

Of course, Φ is a continuously $G\widehat{a}teaux$ differentiable and sequentially weakly lower semi continuous functional whose $G\widehat{a}teaux$ derivative admits a continuous inverse on X^* and Ψ is a continuously $G\widehat{a}teaux$ differentiable functional whose $G\widehat{a}teaux$ derivative is compact. In particular, for each $u, v \in X$ one has

$$\Phi'(u)(v) = \int_0^1 u'(x)v'(x)dx,$$

$$\Psi'(u)(v) = -\int_0^1 f(x, u(x))v(x)dx.$$

Hence, the weak solutions of (1) are exactly the solutions of the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0.$$

Furthermore, thanks to (iii) and to Poincaré inequality, for each $\lambda > 0$, one has that

$$\lim_{||u||\to+\infty}(\Phi(u) + \lambda\Psi(u)) = +\infty.$$

We claim that there exist $r > 0$ and $w \in X$ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < r\frac{(-\Psi(w))}{\Phi(w)}.$$

Now, taking into account that for every $u \in X$, one has

$$\max_{x\in[0,1]}|u(x)| \leq \frac{1}{2}||u||$$

for each $u \in X$, it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) = \sup_{||u||^2\leq 2r}\int_0^1 g(x, u(x))dx \leq \max_{(x,t)\in[0,1]\times[-\sqrt{r/2},\sqrt{r/2}]} g(x, t).$$

Now, thanks to Lemma 2.2, there exist $r > 0$ and $w \in X$ such that $||w||^2 > 2r$ and

$$\max_{(x,t)\in[0,1]\times[-\sqrt{r/2},\sqrt{r/2}]} g(x, t) < 2r\frac{\int_0^1 g(x, w(x))dx}{||w||^2},$$

so our claim is true. Fix ρ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < \rho < r\frac{(-\Psi(w))}{\Phi(w)},$$

and with choose $J = -\Psi$, $x_0 = 0$ and $x_1 = w$, from Proposition 3.1 of [4], we obtain

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

Now, our conclusion follows from Theorem 2.1. □

We now want to point out a consequence of Theorem 2.3:

Theorem 2.4. Let $f : R \to R$ be a continuous function. Put $g(t) = \int_0^t f(\xi)d\xi$ for each $t \in R$ and assume that there exist four positive constants c, d, s and η with $c < \frac{d}{\sqrt{2}}$ and $s < 2$ such that:

(j) $g(t) \geq 0$ for each $t \in [0, d]$,
(jj) $\frac{1}{c^2} \max_{t \in [-c,c]} g(t) < \frac{g(d)}{d^2}$,
(jjj) $g(t) \leq \eta(1 + |t|^s)$ for each $t \in R$.

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (3) admits at least three solutions in X whose norms are less than q.

Remark 2.5. In Theorem 2.4, if $f(t) \geq 0$ for every $t \in [-c, d]$. Then, instead of condition (j) and (jj), we put $\frac{g(c)}{c^2} < \frac{g(d)}{d^2}$, and the result holds.

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