Boundary Control of Viscous Dullin-Gottwald-Holm Equation

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Abstract: this paper investigates the boundary control of viscous DGH equation on [0,1], results the existence of solution of viscous DGH equation in a short time interval with the boundary condition given in the paper and proves the global exponential stablility of DGH equation in $L^2, H^1, H^2, H^3$ with the given boundary condition.

Key words: Dullin-Gottwald-Holm equation;Boundary control;Viscous

1 Introduction

In [1], D.D.Holm and M.F.Staley introduced the b family PDEs that describe the balance between convection and stretching for small viscosity in the dynamics of 1D nonlinear waves in fluids:

$$
\begin{align*}
\frac{m_t}{\text{convection}} + \frac{bu_x m}{\text{stretching}} = \frac{\varepsilon m}{\text{viscosity}}, & \quad u = g * m,
\end{align*}
$$

(1.1)

Here $u = g*m$ denotes $u(x) = \int_{-\infty}^{\infty} g(x-y)m(y)dy$. The convolution relates velocity $u$ to momentum density $m$ by integration against the kernel $g(x)$. [1] studied the effects of the balance parameter $b$ and kernel $g(x)$ on the solitary wave structures and investigated their interactions analytically for $\varepsilon = 0$ and numerically for small viscosity, $\varepsilon \neq 0$.

When (1.1) restrict to the peakon case $g(x) = e^{-|x|/\alpha}$ with length scale $\alpha$, $m = u - \alpha^2 u_{xx}$, the equation (1.1) may be expressed solely in terms of the velocity $u(x,t)$ as

$$
\begin{align*}
u_t + (b + 1)u u_x - \varepsilon u_{xx} = \alpha^2 (u_{xxt} + uu_{xxx} + bu_x u_{xx} - \varepsilon u_{xxxx}) & \quad (1.2)
\end{align*}
$$

For $b = 2$, (1.2) is the one-dimensional version of the three dimensional Navier-Stokes-alpha model for turbulence [2, 3] as $\varepsilon \neq 0$. For $b = 2$, $\varepsilon = 0$, it becomes Camassa-Holm equation [11]. It has a bi-Hamiltionian structure and is completely integrable [12-16]. In [4] Danping Ding, Lixin Tian researched the solution of 1D Navier-Stokes-alpha model or viscosity Camassa-Holm equation and got the existence of the global attractor of viscosity Camassa-Holm equation. For $b = 3$, $\varepsilon = 0$, (1.2) is Degasperis-Procesi equation.

$$
\begin{align*}
u_t - u u_{xx} + 4u u_x = 3u_x u_{xx} + uu_{xxx}, & \quad t > 0, \quad x \in \mathbb{R} 
\end{align*}
$$

(1.3)

It is completely integrable.

Dullin,Gottwald,Holm discussed the following 1+1 quadratically nonlinear equation in this class for a unidirectional water wave with fluid velocity $u(x,t)$.

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\[ m_t + c_0 u_x + um_x + 2mu_x = -\gamma u_{xxx} \]  \hspace{1cm} (1.4)

where \( m = u - \alpha^2 u_{xx} \) is a momentum variable, the constants \( \alpha^2 \) and \( \gamma/c_0 \) are squares of length scales, and \( c_0 = \sqrt{gh} \) is the linear wave speed for undisturbed water at rest at spatial infinity. (1.4) is derived by using asymptotic expansions directly in the Hamiltonian for Euler’s equations in the shallow water regime and thereby shown to be bi-Hamiltonian. It also has a Lax pair formulation. Equation (1.4) combines the linear dispersion of the KdV equation with the nonlinear/nonlocal dispersion of CH equation, yet still preserves integrability via the inverse scattering transform (IST) method. This IST-integrable class of equations contains both the KdV and CH equation as limit cases.

Many researches have been researched on DGH equation. In [9], Lixin Tian, Guilong Gui Yue Liu studied some properties of the solutions of the Cauchy problem and the scattering problem associated to (1.4). The issue of passing to the limit as the dispersive parameter tends to zero for the solution of the DGH equation is investigated, and the convergence of solutions to the DGH equation as \( \alpha^2 \to 0 \) is studied, and the scattering data of the scattering problem for the equation can be explicitly expressed; the new exact peaked solitary wave solutions are obtained in the DGH equation. Besides, Tian and Fan [10] also study the attractor of DP equation which also enlighten the author.

In the recent 20 years, boundary control has been constantly investigated. More people have paid attention to boundary control of KdV, KdVB, MKdV and K-S equation. Byrnes et al. studied local stability of Burgers equation. Van ly et al. consummated this result further but still local. Miroslav Krstic studied global stability of Burgers equation [5]. Biler Rassel Zhang Bingyu studied KdVB equation under periodical boundary condition [7]-[19]. Liu and Kristic studied the stability of KdVB equation in a limited area [6]. Wei-ju Liu, Miroslav Krstic studied global stability of K-S equation [7], Haixia Chao also did some studies on boundary control of K-S equation [8]. Boundary control of DGH equation has not been studied. As above mentioned, the added viscosity is reasonable. In this paper, we study the following viscous DGH equation

\[
\begin{cases}
    u_t - \alpha^2 u_{xxt} - \varepsilon (u - u_{xx})_{xx} + 2\omega u_x + \gamma u_{xxx} + 3uu_x = \alpha^2 (2u_xu_{xx} + uu_{xxx}) \\
    u_0 = u(x, 0) \\
    u(0, t) = u(1, t) \\
    u_x(1, t) = u_x(0, t) \\
    u_{xx}(0, t) = u_{xx}(1, t)
\end{cases}
\]  \hspace{1cm} (1.5)

where \( t > 0, x \in \Omega, \Omega = [0, 1] \). Let \( v = u - \alpha^2 u_{xx} \), (1.5) becomes

\[
\begin{cases}
    (1 - \alpha^2)v_t - \varepsilon v_{xx} + 2\omega v_x + vv_x + 2uv_v = 0 \\
    u_0 = u(x, 0) \\
    u(0, t) = u(1, t) \\
    u_x(1, t) = u_x(0, t) \\
    u_{xx}(0, t) = u_{xx}(1, t)
\end{cases}
\]  \hspace{1cm} (1.6)

where \( t > 0, x \in \Omega, \Omega = [0, 1] \).

In this paper, we discuss the global well-posedness of (1.5). Let \( T > 0, u_0 \in H^3 \), (1.6) admits a unique solution \( u \in \left( (0, T^*), H^3(0, 1) \right) \cap C^0((0, T), H^2(0, 1)) \). The proof is standard; the local well-posedness is established first by using Galerkin Procedure and then the global well-posedness is obtained by finding the needed global a priori estimate.

The outline of this paper is as follows. In section 1, we introduce the differences and relatives among C-H and DGH equation and some relative studies are given out. In section 2, we introduce the main definitions and theorems. In section 3, we establish the proof of the main theorems.

## 2 Main Theorem

The following notation is used in this paper: \((, )\) is the \( L^2 \) inner product and \( \|\cdot\|_{H^m(\Omega)} = \|D^m \cdot\|_{L^2(\Omega)} \). The inner product here is equivalent to the nature inner product in \( H^m(\Omega) \) when \( mes(\Omega) < +\infty \).
\[ \|u\|_{L^2(\Omega)} \Delta |u|, \|Du\|_{L^2(\Omega)} \Delta \|u\|, \|D^m u\|_{L^2(\Omega)} \Delta |D^m u|; \]

\[ B (u, v) = u \nabla v, \text{where } \nabla \text{ is Hamilton operator}. \]

Here we only study one dimensional viscous DGH equation, then \( \nabla u = u_x \).

Denote \( b(u, v, w) = \int_{\Omega} (u \nabla v) w dx \)

With following boundary condition, we have

\[ (B(u, v), w) = -(B(u, w), v) - (B(w, u), v) \]

Then \( (B(u, v), w) = -2(B(u, v), v), (B(u, v), u) = -2(B(v, u), u) \)

Denote \( A = -\Delta, \Delta \text{ is Laplace operator}, v = u + \alpha^2 Au \)

Eq. (1.6) follows that

\[
\begin{align*}
\frac{du}{dt} + \varepsilon Au + 2\omega \nabla u + \gamma \nabla Au + B(u, v) + 2B(v, u) &= 0 \\
u_0 &= u(x, 0) \\
u(0, t) &= u(1, t) \\
u_x(1, t) &= u_x(0, t) \\
u_{xx}(0, t) &= u_{xx}(1, t)
\end{align*}
\]

(2.1)

where \( A \) is a self-adjoint positive operator with compact inverse. Hence the space \( H \) has an orthonormal basis \( \{w_1\}_{j=0}^\infty \) of eigenfunctions of \( A \), i.e. \( Aw_j = \lambda_j w_j \), with \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_j \to \infty \) when \( j \to \infty \)

**Theorem:** With \( u_0 \in H^3 \), Eq.(2.1) has a global solution the solution satisfies the following \( L^2, H^1, H^2, H^3 \) stability:

1. \( \|u_0\|^2 + \alpha^2 \|u_m\|^2 \leq \left( |u_{m0}|^2 + \alpha^2 \|u_{m0}\|^2 \right) \exp(-2\varepsilon \lambda_1 t) \)
2. \( \|u\|^2_{H^2} \leq \left( \|u_0\|^2_{H^2} + \|u_0\|^2_{H^2} \right) \exp(3C_1 + C_2 + C_3 - \varepsilon \lambda_1) \), where \( \varepsilon \in \left( \frac{3C_1 + C_2 + C_3}{\lambda_1}, +\infty \right) \)
3. \( \|u\|^2_{H^3} \leq \left( \|u_0\|^2_{H^3} + \|u_0\|^2_{H^3} \right) \exp\left( \|u_0\|^2_{H^1} \left( \|u_0\|^2_{H^1} + \|u_0\|^2_{H^2} \right) - \varepsilon \lambda_1 \right) t \), where \( \varepsilon \in \left( \|u_0\|^2_{H^1} (\|u_0\|^2_{H^1} + \|u_0\|^2_{H^2}), +\infty \right) \)

3 **The proof of the theorem**

The proof of theorem 1, Galerkin Procedure is used to prove global existence. Let \( \{\phi_j\}_{j=1}^\infty \) be an orthonormal basis of \( H \) consisting of eigenfunctions of the operator \( A \).

The Galerkin Procedure for Eq.2.1 is the ordinary differential system,

\[
\begin{align*}
\frac{du}{dt} + \varepsilon Au_m + 2\omega \nabla u_m + \gamma \nabla Au_m + p_m B(u_m, v_m) + p_m 2B(v_m, u_m) &= 0 \\
u_m(0) &= p_m u(0)
\end{align*}
\]

(3.1a) (3.1b)

where \( v_m = u_m + \alpha^2 Au_m \). Since the nonlinear term is quadratic in \( u_m \), the system (3.1) has a unique solution for a short interval of time \( (0, T_m) \) by the classical theory of ordinary differential equations. Our purpose is to show the existence of the solution under given boundary control.

We take the inner product of (3.1a) with \( u_m \) in \( \Omega \),

\[
\frac{1}{2} \frac{d}{dt} \left( |u_m|^2 + \alpha^2 \|u_m\|^2 \right) + \varepsilon \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 \right) = 0
\]

(3.2)

By poincare inequation,

\[
\frac{d}{dt} \left( |u_m|^2 + \alpha^2 \|u_m\|^2 \right) + 2\varepsilon \lambda_1 \left( |u_m|^2 + \alpha^2 \|u_m\|^2 \right) \leq 0
\]

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By Gronwall inequality,

\[ |u_m|^2 + \alpha^2 \|u_m\|^2 \leq \left( |u_m|^2 + \alpha^2 \|u_m\|^2 \right) \exp (-2\varepsilon_1 t) \leq |u_m|^2 + \alpha^2 \|u_m\|^2 \leq r_1 \]  \tag{3.3}

Integrating (3.3) over the interval \([t, t + r]\)

\[ \int_t^{t+r} \left( \|u_m(s)\|^2 + \alpha^2 |Au_m(s)|^2 \right) ds \leq r_1 \]

Take the inner product of (3.1a) with \(Au_m\) in \((0, 1)\),

\[ \frac{1}{2} \frac{d}{dt} \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 \right) + \varepsilon \left( |Au_m|^2 + \alpha^2 |\nabla Au_m|^2 \right) + p_m (B (u_m, v_m), Au_m) + 2p_m (B (v_m, u_m), Au_m) = 0 \]  \tag{3.4}

Then, we have

\[ |p_m (B (u_m, v_m), Au_m) + 2p_m (B (v_m, u_m), Au_m)| \leq (B (u_m, u_m), Au_m) + (B (Au_m, u_m), Au_m) + (B (u_m, Au_m), Au_m) \]  \tag{3.5}

By Agmon inequality when \(n = 1\):

\[ \|\varphi\| L^\infty \leq c \|\varphi\| L^2 \|\varphi\| H^1 \]

\[ |p_m (B (u_m, Au_m), Au_m)| \leq \frac{1}{2} \|\nabla u_m\| L^\infty \|Au_m\|^2 \leq \frac{c_4}{2} \|u_m\|^{\frac{1}{2}} |Au_m|^{\frac{5}{2}} \]

\[ |p_m (B (Au_m, u_m), Au_m)| \leq \|\nabla u_m\| L^\infty \|Au_m\|^2 \leq c_5 \|u_m\|^{\frac{1}{2}} |Au_m|^{\frac{5}{2}} \]

\[ |p_m (B (u_m, u_m), Au_m)| \leq \frac{1}{2} \|\nabla u_m\| L^\infty \|\nabla u_m\|^2 \leq \frac{c_1}{2} \|u_m\|^{\frac{1}{2}} |Au_m|^{\frac{7}{2}} \]

From above inequalities, we get

\[ |p_m (B (u_m, v_m), Au_m) + 2p_m (B (v_m, u_m), Au_m)| \]

\[ \leq \frac{3c_1}{2} \|u_m\|^{\frac{1}{2}} |Au_m|^{\frac{5}{2}} + \left( \frac{2c_5 + c_4}{2} \right) \|u_m\|^{\frac{1}{2}} |Au_m|^{\frac{5}{2}} \]

\[ \leq \frac{3}{2} c_1 \|u_m\|^{\frac{1}{2}} |Au_m|^{\frac{5}{2}} + \left( \frac{2c_5 + c_4}{2} \right) \|u_m\|^{\frac{1}{2}} |Au_m|^{\frac{5}{2}} \]

\[ \leq c_6 \|u_m\|^{\frac{1}{2}} |Au_m|^{\frac{5}{2}} \left( \|u_m\|^2 + |Au_m|^2 \right) \]

\[ \leq \frac{1}{2} \varepsilon_1 \left( \|u_m\|^2 + |Au_m|^2 \right) + c_7 \|u_m\| |Au_m| \left( \|u_m\|^2 + |Au_m|^2 \right) \]  \tag{3.6}

\[ c_6 = \max (3c_1, 2c_5 + c_4), c_7 = \frac{c_6}{8\varepsilon_1} \]

By Young inequality, it follows that

\[ \frac{1}{2} \frac{d}{dt} \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 \right) + \varepsilon \left( |Au_m|^2 + \alpha^2 |\nabla Au_m|^2 \right) \]

\[ \leq \varepsilon_1 \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 \right) + c_7 \|u_m\| |Au_m| \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 \right) \]  \tag{3.7}

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By Poincare inequality \(|Au_m|^2 > \lambda_1 \|u_m\|^2, |\nabla Au_m|^2 > \lambda_1 |Au_m|^2\) and (3.4), we get

\[
\frac{d}{dt} \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 \right) \leq 2c_7 \|u_m\| |Au_m| \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 \right)
\]

By Young inequality and above inequalities,

\[
\frac{d}{dt} \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 \right) \leq c_7 \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 \right)^2
\]  

(3.8)

Denote \(y = \|u_m(s)\|^2 + \alpha^2 |Au_m(s)|^2\), \(g = c_7 \left( \|u_m(s)\|^2 + \alpha^2 |Au_m(s)|^2 \right)^2\)

By inequality (3.6), we have

\[
\int_t^{t+r} y(s) ds \leq \frac{r_1}{\varepsilon}, \quad \int_t^{t+r} g(s) ds \leq \frac{r_1c_7}{\varepsilon}
\]

By Gronwall's inequality we have

\[
\left( \|u_m\|^2 + \alpha^2 |Au_m|^2 \right) \leq \frac{r_1}{r\varepsilon} \exp \left( \frac{r_1c_7}{\varepsilon} \right) \Delta r, \quad \Delta r > t_0 + r
\]  

(3.9)

where \(r, r_1, c_7\) are nonnegative constants. Integrating (3.7) over the interval \([t, t + r]\)

\[
\varepsilon \int_t^{t+r} \left( |Au_m(s)|^2 + \alpha^2 |\nabla Au_m(s)|^2 \right) ds
\]

\[
\leq \int_t^{t+r} \left( \varepsilon \lambda_1 \left( \|u_m(s)\|^2 + \alpha^2 |Au_m(s)|^2 \right) + \frac{c_7}{2} \left( \|u_m(s)\|^2 + \alpha^2 |Au_m(s)|^2 \right)^2 \right) ds
\]

\[
+ \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 \right)
\]

\[
\leq \left( \varepsilon \lambda_1 r_2 + \frac{c_7 r_3}{2} \right) r + r_3 \Delta r
\]  

(3.10)

Take the inner product of (3.1a) with \(A^2u_m\) in \(\Omega\)

\[
\frac{1}{2} \frac{d}{dt} \left( |Au_m|^2 + \alpha^2 |\nabla Au_m|^2 \right) + \varepsilon \left( |\nabla Au_m|^2 + \alpha^2 |A^2u_m|^2 \right) + p_m \left( B(u_m, v_m), A^2u_m \right) + \frac{1}{2p_m} \left( B(v_m, u_m) \right) \cdot A^2u_m = 0
\]

By Young inequality, we have

\[
\frac{1}{2} \frac{d}{dt} \left( |Au_m|^2 + \alpha^2 |\nabla Au_m|^2 \right) + \varepsilon \left( |\nabla Au_m|^2 + \alpha^2 |A^2u_m|^2 \right)
\]

\[
\leq \varepsilon \lambda_1 \left( |Au_m|^2 + \alpha^2 |\nabla Au_m|^2 \right) + c_8 \|u_m\| |Au_m| \left( |Au_m|^2 + \alpha^2 |\nabla Au_m|^2 \right)
\]  

(3.11)

By Poincare inequality and Young inequality, we have

\[
\frac{d}{dt} \left( |Au_m|^2 + \alpha^2 |\nabla Au_m|^2 \right) \leq c_8 \|u_m\| |Au_m| \left( |Au_m|^2 + \alpha^2 |\nabla Au_m|^2 \right)
\]

\[
\leq \frac{c_8}{2} \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 \right) \left( |Au_m|^2 + \alpha^2 |\nabla Au_m|^2 \right)
\]

From (3.6) and (3.10), we get

\[
\frac{c_8}{2} \int_t^{t+r} \left( \|u_m(s)\|^2 + \alpha^2 |Au_m(s)|^2 \right) ds \leq \frac{c_8 r_1}{2 \varepsilon}
\]

\[
\int_t^{t+r} \left( |Au_m(s)|^2 + \alpha^2 |\nabla Au_m(s)|^2 \right) ds \leq \frac{r_3}{\varepsilon}
\]

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By Gronwall's inequality we have
\[
|Au_m|^2 + \alpha^2 |\nabla Au_m|^2 \leq \frac{r_1}{\varepsilon \varepsilon_5} \exp \left( \frac{c_8 r_1}{2 \varepsilon} \Delta r_5 \right) \quad t > t_0
\]  
(3.12)

Integrating (3.12) over the interval \([t, t + r]\), we get
\[
\varepsilon \int_{t}^{t+r} \left( |\nabla Au_m(s)|^2 + \alpha^2 |A^2 u_m(s)|^2 \right) ds \leq \left( \varepsilon \lambda_1 r_5 + \frac{c_8 r_1^2 r_5}{2} \right) r + r_5
\]

Take the inner product of (3.1a) with \(A^3 u_m\) in \(\Omega\),
\[
\frac{d}{dt} \left( |\nabla Au_m|^2 + \alpha^2 |A^2 u_m|^2 \right) \leq c_9 \|u_m\|^2 |Au_m|^2 + c_{10} |Au_m|^8 |\nabla Au_m|^2
\]

By Gronwall's inequality we have
\[
|\nabla Au_m|^2 + \alpha^2 |A^2 u_m|^2 \leq \left( \frac{r_5}{\varepsilon r} + 2c_10 r_2^2 r_5 r \right) \exp \left( c_9 r_2^2 r \right) \Delta r_6
\]

Now we get: \(|u_m|, \|u_m\|, |Au_m|, |\nabla Au_m|\) and \(|A^2 u_m|\) are finite, then \(|v_m|, \|v_m\|\) and \(|Av_m|\) are finite, then we get \(\frac{du_m}{dt}\) and \(\frac{dv_m}{dt}\) are also finite. By Aubin's compactness Theorem., we conclude that there is a subsequence \(u_m',\) such that \(u_m' \to u\), or equivalently \(v_m' \to v\). Relable \(u_m'\) and \(v_m'\) by \(u_m\) and \(v_m\). Now we prove \(u, v\) satisfy equation (2.1).

Let \(w \in D(A)\) we know \(|w|\) is finite from the above discussion, from ordinary differential equations (3.1a) we get,
\[
(v_m(t), w) + \varepsilon \int_{t_0}^{t} (v_m(s), p_m Aw) ds + 3 \int_{t_0}^{t} (B(v_m(s), u_m(s)), p_m w) ds + \int_{t_0}^{t} (B(u_m(s), v_m(s)), p_m w) ds = (v_m(t_0), w)
\]

Now, it is clear that
\[
\lim_{m \to \infty} \int_{t_0}^{t} (v_m(s), Aw) ds = \int_{t_0}^{t} (v(s), Aw) ds
\]
thus
\[
(v_m(t), w) \to (v(t), w), \int_{t_0}^{t} (v_m(s), p_m A w) ds \to \int_{t_0}^{t} (v(s), Aw) ds \quad m \to \infty
\]

\[
\left| \int_{t_0}^{t} (\nabla v_m(s), p_m w) ds - \int_{t_0}^{t} (\nabla v(s), w) ds \right|
\leq \left| \int_{t_0}^{t} (\nabla v_m(s), p_m w - w) ds \right| + \left| \int_{t_0}^{t} (\nabla v_m(s) - \nabla v(s), w) ds \right|
\leq ||v_m|| \|p_m w - w\| + ||v_m - v\| |w| \to 0
\]

\[
\left| \int_{t_0}^{t} (B(v_m(s), u_m(s)), p_m w) ds - \int_{t_0}^{t} (B(v(s), u(s)), w) ds \right|
\leq I^1_m + I^2_m + I^3_m
\]
where
\[
I^1_m = \left| \int_{t_0}^{t} (B(v_m(s), u_m(s)), p_m w - w) ds \right| \leq \int_{t_0}^{t} |v_m(s)||u_m(s)||p_m w - w| ds \to 0
\]

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\[ I_m^2 = \left| \int_{t_0}^t (B(v_m(s) - v(s), u_m(s), w) \right| \leq \int_{t_0}^t |v_m(s) - v(s)|\|u_m(s)\| |w| \, ds \rightarrow 0 \]

\[ I_m^3 = \left| \int_{t_0}^t (B(v(s), u_m(s) - u(s), w) \right| \leq \int_{t_0}^t |v(s)|\|u_m(s) - u(s)\| |w| \, ds \rightarrow 0 \]

From the above discussion and by Lebesgue theorem we get,

\[ \lim_{m \to \infty} \int_{t_0}^t (B(v_m(s), u_m(s)), p_m w) \, ds = \int_{t_0}^t (B(v(s), u(s)), w) \, ds \]

Similarly, we have

\[ \lim_{m \to \infty} \int_{t_0}^t (B(u_m(s), v_m(s)), p_m w) \, ds = \int_{t_0}^t (B(u(s), v(s)), w) \, ds \]

Above all \( w \in D(A) \) we get

\[ (v(t), w) + \varepsilon \int_{t_0}^t (v(s), Aw) \, ds + 3 \int_{t_0}^t (B(v(s), u(s)), w) \, ds + \int_{t_0}^t (B(u(s), v(s)), w) \, ds = (v(t_0), w) \]

let \( u, v \) be the same two solutions of (2.1), and \( \omega = u - v \), we get

\[ \omega_t - k_{ex} \omega_x - \varepsilon (\omega - l_{ex}) + 3 (u \omega_x + u_x \omega - \omega_x) = \]

\[ = 2 (u_x \omega_x + u_{xx} \omega_x - \omega_x \omega_{xx}) + (u \omega_{xxx} + u_{xxx} \omega - \omega_{xxxx}) \tag{3.13} \]

Take the inner product of (3.13) in [0, 1],

\[ \frac{1}{2} \frac{d}{dt} \left( |\omega|^2 + \|\omega\|^2 \right) + \varepsilon (\|\omega\|^2 + |A\omega|^2) = 2 (u_x, \omega_x) + (u_x, (\omega - \omega_x) \omega_{xx}) \]

by (3.9), we get \( \|u\| \leq r_1 e^{-\varepsilon \lambda t} |Au|^2 < r_2 \)

from (3.5), we get

\[ \frac{d}{dt} \left( |\omega|^2 + \|\omega\|^2 \right) + M \left( |\omega|^2 + \|\omega\|^2 \right) \leq 0 \]

where M is constant then

\[ |\omega|^2 + \|\omega\|^2 \leq \left( |\omega_0|^2 + \|\omega_0\|^2 \right) e^{-Mt} = 0 \]

so \( |\omega| = 0 \) i.e. \( \omega = 0, u = v \)

By (3.3), we have

\[ |u_m|^2 + \alpha^2 |u_m| \leq \left( |u_0|^2 + \alpha^2 |u_0|^2 \right) \exp (-2\varepsilon \lambda t) \]

Now, take the inner product of (2.1) with \( Au \) in (0, 1),

\[ (|Au|^2 + \alpha^2 |\nabla Au|^2) + (B(u, v), Au) + 2 (B(v, u), Au) = 0 \]

By computing, we have

\[ |(B(u, v), Au) + 2 (B(v, u), Au)| = 3 (B(u, u), Au) + (B(Au, u), Au) + (B(u, Au), Au) \]

\[ \|B(u, u), Au\| \leq \left( \|u\|^2 + |Au|^2 \right) \]

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\[(B(Au,u),Au) \leq \|u\| |Au|^2 \leq c_2 (\|u\|^2 + |Au|^2)\]
\[|(B(u,Au),Au)| \leq c_3 |Au|^2 \leq c_3 \left(\|u\|^2 + |Au|^2\right)\]
\[
\frac{1}{2} \frac{d}{dt} \left(\|u\|^2 + |Au|^2\right) \leq (3C_1 + C_2 + C_3 - \varepsilon \lambda_1) \left(\|u\|^2 + |Au|^2\right)
\]
\[\|u\|^2 + |Au|^2 \leq \left(\|u_0\|^2 + |Au_0|^2\right) \exp (3C_1 + C_2 + C_3 - \varepsilon \lambda_1) t\]

So we get Th.(2)

Take the inner product of (2.1) with \(A^2 u\) in (0,1),
\[
\frac{1}{2} \frac{d}{dt} \left(|Au|^2 + |\nabla Au|^2\right) + \varepsilon \left(|\nabla Au|^2 + |A^2 u|^2\right) + (B(u,v),A^2 u) + 2 (B(v,u),A^2 u) = 0
\]

By Young inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \left(|Au|^2 + |\nabla Au|^2\right) + \varepsilon \left(|\nabla Au|^2 + |A^2 u|^2\right)
\leq \left(|Au|^2 + |\nabla Au|^2\right) + \|u\| |Au| \left(|Au|^2 + |\nabla Au|^2\right)
\leq \left(|Au|^2 + |\nabla Au|^2\right) + \|u_0\| \left(\|u_0\|^2 + |Au_0|^2\right) \left(|Au|^2 + |\nabla Au|^2\right)
\]

By Poincare inequality and Young inequality, we have
\[
\frac{d}{dt} \left(|Au|^2 + |\nabla Au|^2\right) \leq \left(\|u_0\| \left(\|u_0\|^2 + |Au_0|^2\right) - \varepsilon \lambda_1\right) \left(|Au|^2 + |\nabla Au|^2\right)
\]
\[|Au|^2 + |\nabla Au|^2 \leq \left(|Au_0|^2 + |\nabla Au_0|^2\right) \exp \left(\|u_0\| \left(\|u_0\|^2 + |Au_0|^2\right) - \varepsilon \lambda_1\right) t\]

We get Th.(3). The proof is completed.

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