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The Combined Laplace-Adomian Method for Handling Singular Integral Equation of Heat Transfer

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Abstract: In this work, we will combine the Laplace transform method with the Adomian decomposition method for analytic treatment of the nonlinear singular integral equation that described heat transfer. The combined algorithm is capable of handling the Abel-type singular equation. The Padé approximants will be used to explore the rapid decay of heat transfer.

Keywords: singular integral equation; heat transfer; Laplace transform method; Adomian decomposition method.

1 Introduction

Volterra examined the nonlinear Volterra integral equation of the form [1–4]

$$u(x) = f(x) + \lambda \int_0^x K(x,t)F(u(t))dt, \tag{1}$$

where $F(u(x))$ is a nonlinear function of the solution $u(x)$, and $K(x,t)$ is the kernel of the integral equation. The Volterra integral equations appear in many physical applications such as neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating.

It is well know that the nonlinear Volterra integral equation (1) is usually handled by series solution methods, such as the Taylor series method and the successive approximations method. The traditional Laplace method by itself cannot be used in this case because of the nonlinearity of this equation. However, it is possible to overcome this difficulty by combining the Laplace transform with the powerful Adomian decomposition method. It is the aim of this work to develop a combined form of the Laplace transform method with the Adomian decomposition method to establish exact solutions or approximate solutions of high degree of accuracy for the nonlinear Volterra integral equations.

2 The combined Laplace-Adomian method

In this work we will assume that the kernel $K(x,t)$ of (1) is a difference kernel that can be expressed by the difference $x-t$. The nonlinear Volterra integral equation (1) can thus be expressed as [4–6]

$$u(x) = f(x) + \lambda \int_0^x K(x-t)F(u(t))\,dt. \tag{2}$$

Applying the Laplace transform to both sides of (2) gives

$$U(s) = \mathcal{L}\{f(x)\} + \lambda\mathcal{L}\{K(x-t)\}\mathcal{L}\{F(u(t))\}. \tag{3}$$

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The Adomian decomposition method and the Adomian polynomials can be used to handle (3) and to address the nonlinear term $F(u(x))$. We first represent the linear term $U(s)$ at the left side by an infinite series of components given by

$$U(s) = \sum_{n=0}^{\infty} U_n(s), \tag{4}$$

and similarly

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{5}$$

where the components $U_n(s), n \geq 0$ will be determined recursively. However, the nonlinear term $F(u(x))$ at the right side of (3) will be represented by an infinite series of the Adomian polynomials A_n in the form

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x), \tag{6}$$

where $A_n, n \geq 0$ are defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i\, u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, \cdots, \tag{7}$$

where the so-called Adomian polynomials A_n can be evaluated for all forms of nonlinearity. In other words, assuming that the nonlinear function is $F(u(x))$, therefore the Adomian polynomials are given by

$$\begin{array}{rcl} A_0 &=& F(u_0), \\ A_1 &=& u_1 F^{'}(u_0), \\ A_2 &=& u_2 F^{'}(u_0) + \frac{1}{2!}u_1^2\, F^{''}(u_0), \\ A_3 &=& u_3 F^{'}(u_0) + u_1 u_2 F^{''}(u_0) + \frac{1}{3!}u_1^3 F^{'''}(u_0), \\ A_4 &=& u_4 F^{'}(u_0) + (\frac{1}{2!}u_2^2 + u_1 u_3)F^{''}(u_0) + \frac{1}{2!}u_1^2 u_2 F^{'''}(u_0) + \frac{1}{4!}u_1^4 F^{(iv)}(u_0). \end{array} \tag{8}$$

Substituting (5) and (6) into (3) leads to

$$\mathcal{L}\{\textstyle\sum_{n=0}^{\infty} U_n(s)\} = \mathcal{L}\{f(x)\} + \lambda\mathcal{L}\{K(x-t)\}\mathcal{L}\{\sum_{n=0}^{\infty} A_n(x)\}. \tag{9}$$

The Adomian decomposition method introduces the recursive relation

$$\begin{array}{rcl} U_0(s) &=& \mathcal{L}\{f(x)\}, \\ U_{k+1}(s) &=& \lambda\mathcal{L}\{K(x-t)\}\mathcal{L}\{A_k(x)\}, k \geq 0. \end{array} \tag{10}$$

Applying the inverse Laplace transform to the first part of (10) gives $u_0(x)$, that will define A_0. Using $A_0(x)$ will enable us to evaluate $u_1(x)$. The determination of $u_0(x)$ and $u_1(x)$ leads to the determination of $A_1(x)$ that will allows us to determine $u_2(x)$, and so on. This in turn will lead to the complete determination of the components of $u_k, k \geq 0$ upon using the second part of (10). The series solution follows immediately after using (5). The obtained series solution may converge to an exact solution if such a solution exists. Otherwise, the series solution can be used for numerical purposes.

The combined Laplace transform Adomian decomposition method for solving nonlinear singular integral equation that governs the heat transfer.

3 The singular integral equation of heat transfer

The model that will be examined is an Abel-type Volterra integral equation that describes the temperature distribution along the surface when the heat transfer to it is balanced by radiation from it.

Lighthill [1,2] presented a nonlinear singular integral equation [2–3] which describes the temperature distribution of the surface of a projectile moving through a laminar layer. The model is given by the nonlinear singular Volterra equation of the second kind

$$u(x) = 1 - \frac{\sqrt{3}}{\pi}\int_0^x \frac{t^{\frac{1}{3}} u^4(t)}{(x-t)^{\frac{2}{3}}} dt, x \in [0, 1], \tag{11}$$

where

$$u(0) = 1, \lim_{t\to\infty} u(t) = 0. \tag{12}$$

By taking Laplace transform of both sides we obtain

$$U(s) = \frac{1}{s} - \frac{\sqrt{3}}{\pi}\mathcal{L}\{x^{-\frac{2}{3}}\}\mathcal{L}\{x^{\frac{1}{3}}u^4(x)\}. \tag{13}$$

The Adomian decomposition method admits the use of

$$\begin{array}{rcl} U_0(s) & = & \frac{1}{s}, \\ U_{k+1}(s)\} & = & -\frac{\sqrt{3}}{\pi}\mathcal{L}\{x^{-\frac{2}{3}}\}\mathcal{L}\{A_k(x)\}, k \geq 0, \end{array} \tag{14}$$

where $A_k(x)$ are the Adomian polynomials for the nonlinear term $u^3(x)$. The Adomian method assumes that the linear term $u(x)$ be decomposed by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{15}$$

and the nonlinear term $x^{\frac{1}{3}}u^4(x)$ by the series

$$x^{\frac{1}{3}}u^4(x) = \sum_{n=0}^{\infty} A_n(x), \tag{16}$$

In what follows we list some of the Adomian polynomials for $x^{\frac{1}{3}}u^4(x)$:

$$\begin{array}{rcl} A_0 & = & x^{\frac{1}{3}}u_0^4, \\ A_1 & = & 4x^{\frac{1}{3}}u_0^3u_1, \\ A_2 & = & 4x^{\frac{1}{3}}u_0^3u_2 + 6x^{\frac{1}{3}}u_0^2u_1^2, \\ A_3 & = & 4x^{\frac{1}{3}}u_0^3u_2 + 12x^{\frac{1}{3}}u_0^2u_1u_2 + 4x^{\frac{1}{3}}u_0u_1^3, \end{array} \tag{17}$$

and so on for other Adomian polynomials. Using the recurrence relation (14), and by using the inverse Laplace transform of U_0, we find

$$u_0(x) = 1. \tag{18}$$

Using this result in U_1 of (14), and using the inverse Laplace transform we find $u_1(x)$. Proceeding in this manner we find the following components

$$\begin{array}{rcl} u_0(x) & = & 1, \\ u_1(x) & = & -1.4609985x^{\frac{2}{3}}, \\ u_2(x) & = & 7.2494161x^{\frac{4}{3}}, \\ u_3(x) & = & -46.449738x^2, \\ u_4(x) & = & 332.75523x^{\frac{8}{3}}, \\ u_5(x) & = & -2536.82x^{\frac{10}{3}}, \\ u_6(x) & = & 20120.06x^4, \\ u_7(x) & = & -163991.85x^{\frac{14}{3}}, \\ u_8(x) & = & 1363564.30x^{\frac{16}{3}}, \\ u_9(x) & = & -11511356x^6. \end{array} \tag{19}$$

The series solution is therefore given by

$$\begin{array}{rcl} u(x) & = & 1 - 1.4609985x^{\frac{2}{3}} + 7.2494161x^{\frac{4}{3}} - 46.449738x^2 \\ & + & 332.75523x^{\frac{8}{3}} - 2536.82x^{\frac{10}{3}} + 20120.06x^4 \\ & - & 163991.85x^{\frac{14}{3}} + 1363564.30x^{\frac{16}{3}} - 11511356x^6 + \cdots. \end{array} \tag{20}$$

It is clear that this series solution satisfies the first condition $u(0) = 1$. It remains to study the rapid decay of the heat transfer of this model. The mathematical structure of $u(x)$ can be studied by using Padé approximants which have the advantage of manipulating the polynomial approximation into a rational function to gain more information about $u(x)$. It is of interest to note that Padé approximants give results with no greater error bounds than approximation by polynomials.

To derive Padé approximants, we first set $t = x^{\frac{2}{3}}$ in the series solution to obtain

$$\begin{aligned} u(t) &= 1 - 1.460998487t + 7.249416142t^2 - 46.44973783t^3 \\ &+ 332.7552332t^4 - 2536.820572t^5 + 20120.06098t^6 \\ &- 163991.8463t^7 + 1363564.301t^8 - 11511356t^9 + \cdots, \end{aligned} \tag{21}$$

Using Maple or Mathematica, The $[3/3]$ and $[4/4]$ Padé approximants are given by

$$[3/3] = \frac{1 + 13.03423334t + 43.02469757t^2 + 24.93153234t^3}{1 + 14.49523183t + 56.95279320t^2 + 49.50724727t^3}, \tag{22}$$

and

$$[4/4] = \frac{1 + 17.75729347t + 98.88920467t^2 + 180.8584715t^3 + 62.90841169t^4}{1 + 19.21829195t + 119.7176840t^2 + 262.8941687t^3 + 139.0424729t^4}, \tag{23}$$

respectively.

Fig 1 below shows the graphs of the Padé approximants $[3/3]$ and $[4/4]$, where the lower graph is for $[4/4]$. The graph explores the rapid decay of heat transfer of this model. This also justifies the second condition $\lim_{t\to\infty} u(t) = 0$.

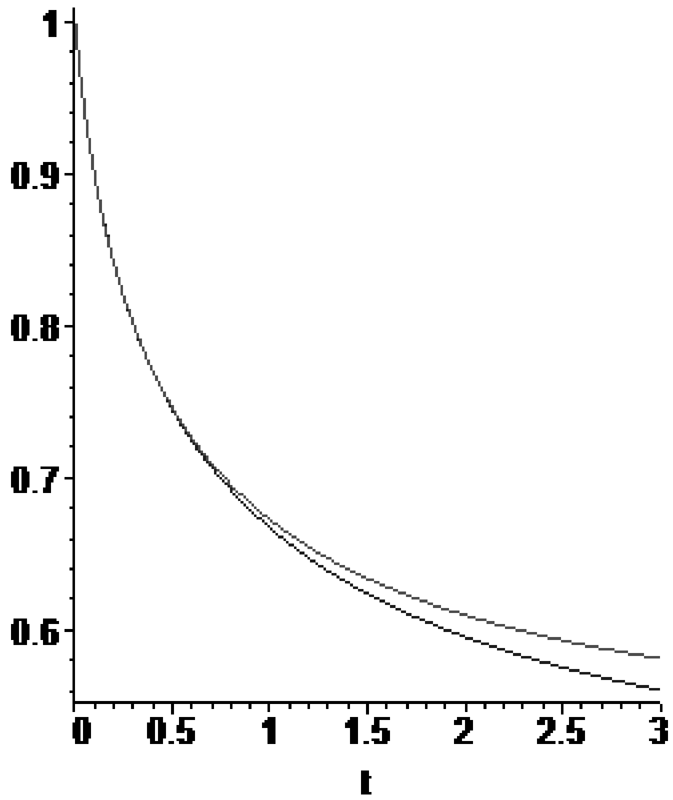

Fig. 1. The Padé approximants $[3,3]$ and $[4/4]$ of $u(t)$, $t = x^{\frac{2}{3}}$.

4 Concluding remarks

A combined form of the Laplace transform method with the Adomian decomposition method is effectively used to handle nonlinear Abel-type integral equation that describes the temperature distribution of the surface of a projectile moving through a laminar layer. The combined Laplace-Adomian method presents a useful way to develop an analytic treatment for these kinds of nonlinear singular integral equations. The proposed scheme can be applied for other nonlinear equations of physics applications.

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