A Hybrid Method for the Generalized Nash Equilibria

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Abstract: A generalized Nash game is an \(N\)–person noncooperative game with nondisjoint strategy sets. In this paper, a new algorithm for solving the generalized Nash equilibria is developed. The algorithm is based on a reformulation of the generalized Nash equilibria as unconstrained optimization. It is proved that the algorithm is globally convergent.

Key words: Generalized Nash equilibria; unconstrained optimization; hybrid method; global convergence

1 Introduction

Game theory is a mathematical theory of socio-economic phenomena exhibiting interaction among decision-makers, called players, whose actions affect each other. The theory has been so far applied in the fields of economics, political science, evolutionary biology, computer science, statistics, accounting, social psychology, law, and branches of philosophy such as epistemology and ethics [2, 6]. The most commonly encountered solution concept in game theory is that of Nash equilibrium (NE) named after John Nash [7]. This notion expresses a kind of optimal collective strategy in a game, where no player has anything to gain by changing only his or her own strategy. The generalized Nash equilibrium (GNE) is an extension of the standard Nash equilibrium, in which each player’s strategy set is dependent on the rival players’ strategies.

We consider the generalized Nash equilibrium problem (GNEP) in this paper. To this end, we first state formally the definition of the Nash equilibrium problem (NEP).

Let \(N\) be the number of players. Each player \(\nu \in \{1, \cdots, N\}\) controls the variables \(x^\nu \in \mathbb{R}^{n^\nu}\). Let \(x = (x^1, \cdots, x^N)^T \in \mathbb{R}^n\) be the vector formed by all these decision variables, where \(n := n_1 + \cdots + n_N\). To emphasize the \(\nu\)th player’s variables within the vector \(x\), we sometimes write \(x = (x^\nu, x^{\nu^-})^T\), where \(x^{\nu^-}\) subsumes all the other players’ variables.

Let \(\theta^\nu : \mathbb{R}^n \to \mathbb{R}\) be the \(\nu\)th player’s payoff function. We assume that these payoff functions are at least continuous, and we further assume that the functions \(\theta^\nu(x) = \theta^\nu(x^\nu, x^{\nu^-})\) are convex in the variable \(x^\nu\). In the standard NEP, the variable \(x^\nu\) belongs to a nonempty, closed and convex set \(X^\nu \subseteq \mathbb{R}^{n^\nu}, \nu = 1, \cdots, N\). Let

\[X := X_1 \times \cdots \times X_N\]

be the Cartesian product of the strategy sets of each player. Then a vector \(x^* \in X\) is called a Nash equilibrium if \(x^{\nu, x^*}\) satisfies

\[\theta^\nu(x^{\nu, x^*}, x^{\nu-, x^*}) \leq \theta^\nu(x^\nu, x^{\nu-, x^*}), \forall x^\nu \in X^\nu\]

for all \(\nu = 1, \cdots, N\).

The GNEP generalizes the situation to some extent since now the strategy sets of player \(\nu\) are allowed to depend on the rival players’ strategies, too. More precisely, we assume that \(X \subseteq \mathbb{R}^n\) is a nonempty, closed
(not necessarily compact) and convex set which represents the joint constraints of all players $\nu = 1, \cdots, N$, so that

$$X_\nu(x^{-\nu}) := \{x^{\nu}|(x^{\nu}, x^{-\nu}) \in X\}$$

becomes the strategy set of player $\nu, \nu = 1, \cdots, N$. Note that our assumptions on $X$ imply that each set $X_\nu(x^{-\nu})$ is also closed and convex.

In the context of GNEPs, we also need the set

$$\Omega(x) := X_1(x^{-1}) \times \cdots \times X_N(x^{-N}).$$

Then a vector $x^* \in \Omega(x^*)$ is called a generalized Nash equilibrium if $x^{*,\nu}$ satisfies

$$\theta_\nu(x^{*,\nu}, x^{*,-\nu}) \leq \theta_\nu(x^{\nu}, x^{*,-\nu}), \forall x^{\nu} \in X_\nu(x^{*,-\nu})$$

for all $\nu = 1, \cdots, N$.

It is by now a well-known fact that the Nash equilibrium problem in which each player solves a convex program can be formulated and solved as a finite-dimensional variational inequality (VI), to which a host of computational methods are applicable [3]. The connection between the generalized Nash games and quasi-variational inequalities (QVIs) was recognized by Bensoussan [1] as early as 1974 who studied these problems with quadratic functionals in a Hilbert space. Harker [4] revised these problems in Euclidean spaces. Robinson [9, 10] discussed an application of a generalized Nash problem in a two-sided game model of combat. Kocvara and Outrata [5] discussed a class of QVIs with applications to engineering. Wei and Smeers [12] introduced a QVI formulation of a spatial oligopolistic electricity model with Cournot generators and regulated transmission prices. Pang [8] recently analyzed the computational resolution of the generalized Nash game by a penalization method for the noncooperative multi-leader-follower games. Using a regularized Nikaido-Isoda-function, the authors of [11] presented three optimization problems related to the generalized Nash equilibrium problem.

Computing a generalized Nash equilibrium remains a challenging task up-to-date. As such, it is interesting to develop efficient computational methods for solving a GNEP. In this paper, based on the reformulation of GNEP as an unconstrained minimization problem given by [11], a new hybrid algorithm for the GNEP is developed. It is proved that the algorithm is globally convergent.

## 2 GNEP as Unconstrained Minimization

First, we state the following Nikaido-Isoda-function:

$$\Psi(x, y) := \sum_{\nu=1}^{N} \left[ \theta_\nu(x^{\nu}, x^{-\nu}) - \theta_\nu(y^{\nu}, x^{-\nu}) \right], \quad (1)$$

and a definition which we have taken from [11].

**Definition 2.1** A vector $x^* \in X$ is called a normalized Nash equilibrium of the GNEP, if $\sup_{y \in X} \Psi(x^*, y) \leq 0$ holds, where $\Psi$ denotes the Nikaido-Isoda-function from (1).

It is not difficult to see that a normalized Nash equilibrium is always a solution of the GNEP, whereas the converse is not true in general.

Let $0 < \alpha < \beta$ be two given parameters, let

$$\Psi_\alpha(x, y) := \sum_{\nu=1}^{N} \left[ \theta_\nu(x^{\nu}, x^{-\nu}) - \theta_\nu(y^{\nu}, x^{-\nu}) - \frac{\alpha}{2} \|x^{\nu} - y^{\nu}\|^2 \right],$$

$$\Psi_\beta(x, y) := \sum_{\nu=1}^{N} \left[ \theta_\nu(x^{\nu}, x^{-\nu}) - \theta_\nu(y^{\nu}, x^{-\nu}) - \frac{\beta}{2} \|x^{\nu} - y^{\nu}\|^2 \right],$$

$$\hat{V}_\alpha(x) := \max_{y \in X} \Psi_\alpha(x, y) = \Psi_\alpha(x, \hat{y}_\alpha(x)),$$
\[ \hat{V}_\beta(x) := \max_{y \in X} \Psi_\beta(x, y) = \Psi_\beta(x, \hat{y}_\beta(x)), \]

where \( \hat{y}_\alpha(x) = \arg \max_{y \in X} [\Psi(x, y) - \frac{\theta}{2} \|x - y\|^2] \) and \( \hat{y}_\beta(x) = \arg \max_{y \in X} [\Psi(x, y) - \frac{\lambda}{2} \|x - y\|^2] \).

Define
\[ \hat{V}_{\alpha\beta}(x) := \hat{V}_\alpha(x) - \hat{V}_\beta(x), \quad x \in \mathbb{R}^n. \]

The following two important conclusions were also given by [11].

**Theorem 2.2** Let \( x^* \) be a stationary point of \( \hat{V}_{\alpha\beta} \). If Assumption 2.1 holds at \( x = x^* \), then \( x^* \) is a normalized Nash equilibrium of the GNEP.

**Remark 2.1** Theorem 2.1 shows that the normalized Nash equilibria of the GNEP are precisely the global minima of the unconstrained optimization problem
\[ \min_{x \in \mathbb{R}^n} \hat{V}_{\alpha\beta}(x). \]

### 3 A Hybrid Algorithm and Its Convergence

Due to the results of the previous sections, we present a hybrid algorithm and show that it is globally convergent. In this section, we assume that the payoff functions \( \theta_\nu(x) = \theta_\nu(x^\nu, x^{-\nu}) \) are convex with respect to \( x^{-\nu} \) and continuously differentiable for each player \( \nu = 1, \ldots, N \).

Now we formally state our algorithm as follows.

**Algorithm 1.**

Step 0. Choose \( x^0 \in \mathbb{R}^n, \omega \in (0, 1), \zeta \in (0, 1), \delta \in (0, 1), \sigma \in (0, 1), \) and sufficiently small \( \epsilon \geq 0 \). Let \( k := 0 \).

Step 1. If \( \hat{V}_{\alpha\beta}(x^k) \leq \epsilon \) or \( \|\nabla \hat{V}_{\alpha\beta}(x^k)\| \leq \epsilon \), stop.

Step 2. Select an arbitrary vector \( d^k \) in \( \mathbb{R}^n \). If
\[ \hat{V}_{\alpha\beta}(x^k + d^k) \leq \zeta \hat{V}_{\alpha\beta}(x^k), \]
then let \( \lambda_k := 1 \) and go to Step 4. Otherwise, if \( d^k \) does not satisfy the condition
\[ \langle \nabla \hat{V}_{\alpha\beta}(x^k), d^k \rangle \leq -\sigma \max\{\|\nabla \hat{V}_{\alpha\beta}(x^k)\|^2, \|d^k\|^2\}, \]
then set \( d^k := -\nabla \hat{V}_{\alpha\beta}(x^k) \).

Step 3. Find the smallest nonnegative integer \( m_k \) satisfying
\[ \hat{V}_{\alpha\beta}(x^k + \omega^{m_k} d^k) - \hat{V}_{\alpha\beta}(x^k) \leq \delta \omega^{m_k} \langle \nabla \hat{V}_{\alpha\beta}(x^k), d^k \rangle \]
and let \( \lambda_k := \omega^{m_k} \).

Step 4. Set \( x^{k+1} := x^k + \lambda_k d^k \) and \( k := k + 1 \). Go to Step 1.

It is easy to see that the Algorithm 1 is well defined.

Now, we consider the global convergence of Algorithm 1.
**Theorem 3.1** Let $\epsilon = 0$ and suppose that the Algorithm 1 generates an infinite sequence $\{x^k\}$. Let $D$ be a closed convex set which contains an infinite subsequence $\{x^k\}_{k \in N_0}$, where $N_0 \subseteq N = \{1, 2, \ldots\}$. If $\nabla V_{\alpha \beta}(x)$ is uniformly continuous on $D$.

Then, either

\[ \lim_{k \to \infty} \hat{V}_{\alpha \beta}(x^k) = 0 \]

or else,

\[ \lim_{k \to \infty, k \in N_0} \nabla V_{\alpha \beta}(x^k) = 0 \]

**Proof.** Since the sequence $\{\hat{V}_{\alpha \beta}(x^k)\}$ is nonnegative and decreases monotonically, it converges to some $\hat{V}_{\alpha \beta}^*$. If $\lim_{k \to \infty} \hat{V}_{\alpha \beta}(x^k) = \hat{V}_{\alpha \beta}^* = 0$, then this theorem is proved.

If $\lim_{k \to \infty} \hat{V}_{\alpha \beta}(x^k) = \hat{V}_{\alpha \beta}^* > 0$, we show that

\[ \lim_{k \to \infty, k \in N_0} \nabla V_{\alpha \beta}(x^k) = 0 \]

by contradiction. Assume that there exists an $\epsilon_1 > 0$ and a subsequence $\{x^k\}_{k \in N_1}$, where $N_1 \subseteq N_0$, such that

\[ \|\nabla \hat{V}_{\alpha \beta}(x^k)\| \geq \epsilon_1, \quad \forall k \in N_1. \]  \hspace{1cm} (5)

In this case, it is easy to know that $d^k$ satisfies (2) only for finitely many $k$. Without loss of generality, we assume that (2) does not hold for all $k$. Then, by construction of Algorithm 1, condition (3) is satisfied for all $k$. So it follows from (5) and (3) that

\[ \langle \nabla \hat{V}_{\alpha \beta}(x^k), d^k \rangle \leq -\sigma \epsilon_1^2 < 0, \quad \forall k \in N_1. \]  \hspace{1cm} (6)

Since $\{\hat{V}_{\alpha \beta}(x^k)\}$ is decreasing and nonnegative, the line search rule (4) implies that

\[ \lim_{k \in N_1, k \to \infty} \lambda_k \langle \nabla \hat{V}_{\alpha \beta}(x^k), d^k \rangle = 0, \]

which, together with (6), yields

\[ \lim_{k \in N_1, k \to \infty} \lambda_k = 0. \]  \hspace{1cm} (7)

Form the Armijo-type line search rule (4) and (7), we know $\omega^{-1} \lambda_k$ must violate inequality (4) for all $k \in N_1$ sufficiently large, i.e.,

\[ \hat{V}_{\alpha \beta}(x^k + \omega^{-1} \lambda_k d^k) - \hat{V}_{\alpha \beta}(x^k) > \delta \omega^{-1} \lambda_k \langle \nabla \hat{V}_{\alpha \beta}(x^k), d^k \rangle. \]

By mean value theorem, we have

\[ \langle \nabla \hat{V}_{\alpha \beta}(z^k), d^k \rangle > \delta \langle \nabla \hat{V}_{\alpha \beta}(x^k), d^k \rangle, \]

where, $z^k = x^k + \theta_k \omega^{-1} \lambda_k d^k$ and $\theta_k \in (0, 1)$. It follows from this inequality that

\[ \langle \nabla \hat{V}_{\alpha \beta}(z^k) - \nabla \hat{V}_{\alpha \beta}(x^k), d^k \rangle > (\delta - 1) \langle \nabla \hat{V}_{\alpha \beta}(x^k), d^k \rangle, \]

which, together with Cauchy-Schwarz inequality, deduces that

\[ \|\nabla \hat{V}_{\alpha \beta}(z^k) - \nabla \hat{V}_{\alpha \beta}(x^k)\| \geq \frac{1}{\|d^k\|} \langle \nabla \hat{V}_{\alpha \beta}(z^k) - \nabla \hat{V}_{\alpha \beta}(x^k), d^k \rangle \]

\[ > \delta \frac{1 - 1}{\|d^k\|} \langle \nabla \hat{V}_{\alpha \beta}(x^k), d^k \rangle \]  \hspace{1cm} (8)

Moreover, from (3) and (4), we can also get

\[ \hat{V}_{\alpha \beta}(x^k) - \hat{V}_{\alpha \beta}(x^{k+1}) \geq \delta \sigma \max \{\lambda_k \|\nabla \hat{V}_{\alpha \beta}(x^k)\|, \lambda_k \|d^k\|^2\} \]

\[ \geq \delta \sigma \lambda_k \|d^k\|^2. \]  \hspace{1cm} (9)

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Since the sequence $\{\hat{V}_{\alpha\beta}(x^k)\}$ is nonnegative and decreases monotonically, from (9) and (7), we have
\[
\lim_{k \in N_1, k \to \infty} \lambda_k d^k = 0,
\]
which implies that
\[
\lim_{k \in N_1, k \to \infty} \|z^k - x^k\| = 0.
\] (10)
Thus, from the assumption and (10), we get
\[
\lim_{k \in N_1, k \to \infty} \|\nabla \hat{V}_{\alpha\beta}(z^k) - \nabla \hat{V}_{\alpha\beta}(x^k)\| = 0.
\] (11)
On the other hand, from (5), (3) and (8), we have
\[
\|\nabla \hat{V}_{\alpha\beta}(z^k) - \nabla \hat{V}_{\alpha\beta}(x^k)\| \geq (1 - \delta)\sigma \max \{ \frac{\|\nabla \hat{V}_{\alpha\beta}(x^k)\|^2}{\|d^k\|}, \|d^k\| \} \geq (1 - \delta)\sigma \max \{ \frac{\epsilon_1^2}{\|d^k\|}, \|d^k\| \}. \] (12)
From (11) and (12), we have
\[
\lim_{k \in N_1, k \to \infty} \frac{\epsilon_1}{\|d^k\|} = 0, \quad \lim_{k \in N_1, k \to \infty} \|d^k\| = 0.
\]
It is obvious that the above two equalities are contradictious. This completes the proof of the theorem. □

If $x^*$ is an accumulation point of the sequence $\{x^k\}$, then there exists a subsequence $\{x^k\}_{k \in K}$ which converges to $x^*$. So the subsequence $\{x^k\}_{k \in K}$ is bounded. There exists an bounded and closed convex set $D$ which contains the subsequence $\{x^k\}_{k \in K}$. Since $\nabla \hat{V}_{\alpha\beta}$ is continuous, $\nabla \hat{V}_{\alpha\beta}(x^k)$ is uniformly continuous on $D$. For this reason, we obtain a corollary to Theorem 3.1.

**Corollary 3.1** Let $\epsilon = 0$ and suppose that the Algorithm 1 generates an infinite sequence $\{x^k\}$. Then any accumulation point $x^*$ of the sequence $\{x^k\}$ is a stationary point of the function $\hat{V}_{\alpha\beta}$.

Using Corollary 3.1 and Theorem 2.2 in section 2, we obtain the following theorem.

**Theorem 3.2** If $x^*$ is an accumulation point of the sequence $\{x^k\}$ and Assumption 2.1 holds at $x = x^*$, then $x^*$ is a normalized Nash equilibrium of the GNEP.

**Remark 3.1** The direction $d^k$ in (2) is important to the Algorithm 1, which is related to the efficiency of it. To obtain the fast convergence rate, we need to find a good direction $d^k$. So the choice of the direction $d^k$ is an interesting topic for further discussion.

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**References**


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