Well-posedness for a New Completely Integrable Shallow Water Wave Equation

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Abstract: Recently a kind of nonlinear dispersive shallow water equation has been researched. In this paper, we investigate well-posedness in classes of discontinues functions for the nonlinear and third order dispersive Degasperis-Procesi with dispersive term

\[ u_t - u_{txx} + 4u u_x + \gamma(u - u_{xx})_x = 3u_x u_{xx} + uu_{xxx}. \]

By using viscous approximations and prior estimates, we prove existence of at least one weak solution, satisfying a restricted set of entropy inequality in the class \( L^2(R) \cap L^4(R) \). And in \( L^1(R) \cap BV(R) \) there exists entropy weak solution.

Key words: weak solution; entropy weak solution; priori estimates; well-posedness

1 Introduction

In ([1]), Degasperis and Proesi studied the following family of third order dispersive PDE conservation laws,

\[ u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xx} = \left( c_1 u^2 + c_2 u_x^2 + c_3 uu_{xx} \right)_x \]

where \( \alpha, c_0, c_1, c_2 \) and \( c_3 \) are real constants. They found that there are at least four equations that satisfy the completely integrability condition within this family: KdV equation, Camassa-Holm equation, Dullin-Gottwald-Holm equation and Degasperis-Procesi equation.

With \( \alpha = c_2 = c_3 = 0 \) in Eq(1), it becomes the well-known Korteweg-de Veris equation. The KdV equation is completely integrable and its solitary waves are solitions ([2],[3]). The Cauchy problem of the KdV equation has been the subject of many studies, and a satisfactory local or global existence theory is now in hand ([4]).

For \( c_1 = -\frac{3}{2}c_3/\alpha^2, c_2 = c_3/2 \), Eq(1) becomes the Camassa-Holm equation, it has a bi-Hamiltonian structure and is completely integrable ([5]). In ([6]), Dangping Ding and Lixin Tian researched solution of dissipative Camassa-Holm equation in total space. Tian, Song, Yin ([7],[8]) considered the generalized Camassa-Holm equation and derived some new exact peakon and compacton.

Dullin, Gottwald, Holm ([9]) discussed the following 1+1 quadratically nonlinear equation in this class for an unidirectional water wave with fluid velocity \( u(x,t) \),

\[ m_t + c_0 u_x + um_x + 2mu_x = -\gamma u_{xxx}, \quad x \in R, \quad t \in R. \]

In ([10],[11]) Lixin Tian, Guilong Gui and Yue Liu studied the well-posedness of the Cauchy problem and the scattering problem for DGH equation.

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With \( c_1 = -2 c_3 / \alpha^2 \), \( c_2 = c_3 \) in Eq(1), we find the Degasperis-Procesi equation of the form

\[
 u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \quad t > 0, \quad x \in \mathbb{R}.
\]  

(3)

Degasperis, Holm and Hone ([12]) proved the integrability of Eq(3) by constructing a Lax pair. They also showed that Eq(3) has bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admits exact peakon solutions which are analogous to the Camassa-Holm equation. After the Degasperis-Procesi equation(3) was derived, many papers were devoted to its study. For example, Yin proved local well-posedness for Eq(3) with initial data \( u_0 \in H^s (\mathbb{R}) \), \( (s > 3/2) \) ([13]) and derived the precise blow-up scenario and a blow-up result. The global existence of strong solutions and global weak solutions to Eq(3) was also investigated in ([14],[15],[16],[17]). Recently Lenells ([18]) has classified all weak traveling wave solutions. Matsuno ([19]) studied multi-soliton solutions and their peakon limit. Coclite and Karlsen ([20]) proved there exists a unique global entropy weak solution in \( L^1 (\mathbb{R}) \cap BV (\mathbb{R}) \) and \( L^2 (\mathbb{R}) \cap L^4 (\mathbb{R}) \).

Eq(3) with a strong dispersive term, we get

\[
 u_t - u_{txx} + 4uu_x + \gamma (u - u_{xx})_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{R}
\]  

(4)

Considering the Cauchy problem of Eq(4) with initial data \( u_0 \), it is equivalent to the hyperbolic-elliptic system

\[
 \begin{cases}
 \partial_t u + \partial_x (\frac{u^2}{2} + \gamma u) + \partial_x P = 0, \quad (t, x) \in R_+ \times R, \\
 -\partial^2_{xx} P + P = \frac{3}{2} u^2, \quad (t, x) \in R_+ \times R, \\
 u(0, x) = u_0(x), \quad x \in R.
 \end{cases}
\]

or

\[
 \begin{cases}
 \partial_t u + \partial_x [\frac{u^2}{2} + \gamma u + G_1 \ast (\frac{3}{2} u^2)] = 0, \quad (t, x) \in R_+ \times R, \\
 u(0, x) = u_0(x), \quad x \in R.
 \end{cases}
\]  

(5)

For any \( \lambda > 0 \), the operator \( (\lambda^2 - \partial^2_{xx})^{-1} \) has a convolution structure:

\[
 (\lambda^2 - \partial^2_{xx})^{-1}(f)(x) = (G_\lambda \ast f)(x) = \frac{1}{2\lambda} \int_R e^{-|x-y|/\lambda \lambda} f(y) dy, \quad x \in R
\]

where \( G_\lambda := \frac{1}{\sqrt{2\pi}} e^{-\lambda |x|} \). Hence we have \( P(t, x) = P^u(t, x) = G_1 \ast (\frac{3}{2} u^2)(t, x) \).

The conservation laws of Eq(3) is the same with Degasperis-Procesi equation.

We shall use the following notations without further comment. “*” standard for the convolution. We use (,) to present the standard inner product in \( L^2 \), for \( 1 \leq p \leq \infty \). The norm in the Lebesgue space \( L^p \) will be written \( \| . \|_L^p \), while \( \| . \|_s, s \geq 0 \) will stand for the norm in the classical Sobolev space \( H^s (\mathbb{R}) \). \( \| \cdot \|_X \) is defined as the norm in Banach space \( X \).

2

Viscous approximations and a priori estimates

We will prove existence of a solution to the Cauchy problem (3) by analyzing the limiting behavior of a sequence of smooth functions \( \{u_\varepsilon \}_{\varepsilon > 0} \), where each function \( u_\varepsilon \) solves the following viscous problem:

\[
 \begin{cases}
 \partial_t u_\varepsilon - \partial^3_{xxxx} u_\varepsilon + 4u_\varepsilon \partial_x u_\varepsilon + \gamma \partial_x (u_\varepsilon - \partial^2_{xx} u_\varepsilon) \\
 = 3\partial_x u_\varepsilon \partial^2_{xx} u_\varepsilon + u_\varepsilon \partial^3_{xxx} u_\varepsilon + \varepsilon \partial^4_{xxxx} u_\varepsilon, \quad (t, x) \in R^+ \times R, \\
 u_\varepsilon (0, x) = u_{0, \varepsilon}(x), \quad x \in R.
 \end{cases}
\]  

(6)

This problem can be stated equivalently as a parabolic–elliptic system:

\[
 \begin{cases}
 \partial_t u_\varepsilon + \partial_x \left( \frac{u^2_\varepsilon}{2} + \gamma u_\varepsilon \right) + \partial_x P_\varepsilon = \varepsilon \partial^2_{xx} u_\varepsilon, \quad (t, x) \in R^+ \times R, \\
 -\partial^2_{xx} P_\varepsilon + P_\varepsilon = \frac{3}{2} u^2_\varepsilon, \quad (t, x) \in R^+ \times R, \\
 u_\varepsilon (0, x) = u_{0, \varepsilon}(x), \quad x \in R.
 \end{cases}
\]  

(7)

where \( P_\varepsilon(t, x) = p^u_\varepsilon(t, x) = G_1 \ast (\frac{3}{2} u^2_\varepsilon)(t, x) = \frac{3}{4} \int_R e^{-|x-y|} (u_\varepsilon(t,y))^2 dy.\)
To begin with, we assume that
\[ u_0 \in L^2(R), \]
and
\[ u_{0,\epsilon} \in H^1(R), \quad t \geq 2, \quad \|u_{0,\epsilon}\|_{L^2(R)} \leq \|u_0\|_{L^2(R)}, u_{0,\epsilon} \to u_0 \quad \text{in} \quad L^2(R). \]  

We will impose additional conditions on the initial data as we make progress.

We begin by stating a lemma which shows that the viscous problem (6) is wellposed for each fixed \( \epsilon > 0 \).

**Lemma 1** ([21], Theorem 2.3) Assume (8) and (9) hold, and fix any \( \epsilon > 0 \). Then there exists a unique global smooth solution \( u_\epsilon = u_\epsilon(t,x) \) to the Cauchy Problem (7) belonging to \( C([0,\infty); H^1(R)) \).

### 2.1 \( L^2 \) estimates and some consequences

Next we prove a uniform \( L^2 \) bound on the approximate solution \( u_\epsilon \), which reinforces the whole analysis in this paper.

**Lemma 2** (Energy estimate). Assume (8) and (9) hold, and fix any \( \epsilon > 0 \). The following bounds hold for any \( t \geq 0 \):
\[
\|u_\epsilon(t,\cdot)\|_{L^2(R)} \leq 2 \sqrt{2} \|u_0\|_{L^2(R)} + \sqrt{\epsilon} \|\partial_x u_\epsilon\|_{L^2(R \times \mathbb{R})} \leq 2 \|u_0\|_{L^2(R)}.
\]

For the proof of this lemma we introduce the quantity \( v_\epsilon = v_\epsilon(t,x) \) defined by
\[
v_\epsilon(t,x) = (G_2 \ast u_\epsilon)(t,x) = \int_R e^{-2|x-y|} u_\epsilon(t,y) dy, \quad t \geq 0, \quad x \in R.
\]

Since \( G_2(x) = e^{-2|x|} \) is Green’s function of the operator \( 4 - \partial_{xx}^2 \), we see that \( v_\epsilon \) also satisfies the equation
\[
-\partial_{xx}^2 v_\epsilon + 4v_\epsilon = u_\epsilon.
\]  
(10)

To prove Lemma 2 we shall need the following estimates on \( v_\epsilon \):

**Lemma 3** Assume (8) and (9) hold, and fix any \( \epsilon > 0 \). Then the following identity holds for any \( t \geq 0 \):
\[
\|\partial_{xx}^2 v_\epsilon(t,\cdot)\|_{L^2(R)}^2 + 5 \|\partial_x v_\epsilon(t,\cdot)\|_{L^2(R)}^2 + 4 \|v_\epsilon(t,\cdot)\|_{L^2(R)}^2 + 2 \epsilon \int^t_0 \|\partial_{xx}^3 v_\epsilon(\tau,\cdot)\|_{L^2(R)}^2 + 5 \|\partial_{xx}^2 v_\epsilon(\tau,\cdot)\|_{L^2(R)}^2 + 4 \|\partial_x v_\epsilon(\tau,\cdot)\|_{L^2(R)}^2 d\tau \leq \|v_\epsilon(0,\cdot)\|_{L^2(R)}^2 + 5 \|\partial_x v_\epsilon(0,\cdot)\|_{L^2(R)}^2 + 4 \|v_\epsilon(0,\cdot)\|_{L^2(R)}^2.
\]  
(11)

**Proof.** Multiplying the first equation of (7) by \( v_\epsilon - \partial_{xx}^2 v_\epsilon \) and integrating over \( R \), we get
\[
\int_R \partial_t u_\epsilon(v_\epsilon - \partial_{xx}^2 v_\epsilon) dx - \epsilon \int_R \partial_{xx}^2 u_\epsilon(v_\epsilon - \partial_{xx}^2 v_\epsilon) dx = -\int_R u_\epsilon \partial_x u_\epsilon(v_\epsilon - \partial_{xx}^2 v_\epsilon) dx - \int_R \partial_x P_{\epsilon}(v_\epsilon - \partial_{xx}^2 v_\epsilon) dx - \gamma \int_R \partial_x u_\epsilon(v_\epsilon - \partial_{xx}^2 v_\epsilon) dx.
\]  
(12)

For the left-hand side of this identity, using (10), we have
\[
\int_R \partial_t u_\epsilon(v_\epsilon - \partial_{xx}^2 v_\epsilon) dx - \epsilon \int_R \partial_{xx}^2 u_\epsilon(v_\epsilon - \partial_{xx}^2 v_\epsilon) dx = \frac{1}{2} \frac{d}{dt} \int_R (4 v_\epsilon^2 + 5 (\partial_x v_\epsilon)^2 + (\partial_{xx}^2 v_\epsilon)^2) dx + \epsilon \int_R (4 (\partial_x v_\epsilon)^2 + 5 (\partial_{xx}^2 v_\epsilon)^2 + (\partial_{xxx}^3 v_\epsilon)^2) dx.
\]  
(13)
where we have used (7), (10), and integration-by-parts.

\[- \int_R u_\varepsilon \partial_x u_\varepsilon (v_\varepsilon - \partial_x^2 v_\varepsilon) \, dx - \int_R \partial_x P_\varepsilon (v_\varepsilon - \partial_x^2 v_\varepsilon) \, dx = - \int_R u_\varepsilon \partial_x u_\varepsilon (4v_\varepsilon - \partial_x^2 v_\varepsilon) \, dx = 0, \quad (14)\]

\[-\gamma \int_R \partial_x u_\varepsilon (v_\varepsilon - \partial_x^2 v_\varepsilon) \, dx = -\gamma \int_R (4\partial_x v_\varepsilon - \partial_x^3 v_\varepsilon)(v_\varepsilon - \partial_x^2 v_\varepsilon) \, dx = 0.\]

Substituting (13) and (14) into (12) yields

\[\frac{d}{dt} \int_R (4v_\varepsilon^2 + 5(\partial_x v_\varepsilon)^2 + (\partial_x^2 v_\varepsilon)^2) \, dx + 2\varepsilon \int_R (4(\partial_x v_\varepsilon)^2 + 5(\partial_x^2 v_\varepsilon)^2 + (\partial_x^3 v_\varepsilon)^2) \, dx = 0.\]

Integrating this inequality over \([0, t]\), we obtain (11).

**Proof of Lemma 2.** We omit the proof since it is similar to the one found in [20].

**Lemma 4** ([20]) Assume (8) and (9) hold, and fix any \(\varepsilon > 0\). Then

\[P_\varepsilon \geq 0,\]

\[\|P_\varepsilon(t, \cdot)\|_{L^1(R)}, \|\partial_x P_\varepsilon(t, \cdot)\|_{L^1(R)} \leq 12 \|u_0\|^2_{L^2}, t \geq 0 \quad (15)\]

\[\|P_\varepsilon\|_{L^\infty(R_+ \times R)}, \|\partial_x P_\varepsilon\|_{L^\infty(R_+ \times R)} \leq 6 \|u_0\|^2_{L^2}, \quad (16)\]

\[\left\|\partial_x^2 P_\varepsilon(t, \cdot)\right\|_{L^1(R)} \leq 24 \|u_0\|^2_{L^2}, t \geq 0.\]

### 2.2 \(L^4\) Estimate

Next we prove that the viscous approximations are uniformly bounded in \(L^4\). For this purpose, we need to assume, in addition to (8) and (9),

\[u_0, u_0, \varepsilon \in L^4(R), \|u_0, \varepsilon\|_{L^4(R)} \leq \|u_0\|_{L^4(R)}. \quad (17)\]

**Lemma 5** ([20]) Assume (8), (9) and (17) hold, and fix any \(\varepsilon > 0\). Then for any \(t \geq 0\)

\[\|u_\varepsilon(t, \cdot)\|^4_{L^4(R)} \leq e^{12\|u_0\|^2_{L^2(R)}t} \|u_0\|^4_{L^4(R)} + 8 \|u_0\|^2_{L^2(R)} (e^{12\|u_0\|^2_{L^2(R)}t} - 1).\]

### 2.3 \(L^1\) Estimate

As a consequence of the \(L^2\) bound in Lemma 2, we can bound \(u_\varepsilon\) in \(L^1\), we assume in addition to (8) and (9),

\[u_0, u_0, \varepsilon \in L^1(R), \|u_0, \varepsilon\|_{L^1(R)} \leq \|u_0\|_{L^1(R)}. \quad (18)\]

**Lemma 6** \((L^1\text{-estimate} [20])\) Assume (8), (9), and (18) hold, and fix any \(\varepsilon > 0\). Then

\[\|u_\varepsilon(t, \cdot)\|_{L^1(R)} \leq \|u_0\|_{L^1(R)} + 12t \|u_0\|^2_{L^2(R)}, t \geq 0. \quad (19)\]

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2.4 BV and $L^\infty$ estimates

In this subsection we prove that the sequence $\{u_\varepsilon\}_{\varepsilon > 0}$ is bounded in BV. To this end, we assume in addition to (8) and (9),

$$u_0, u_{0, \varepsilon} \in BV(R), |u_{0, \varepsilon}|_{BV(R)} \leq |u_0|_{BV(R)}. \tag{20}$$

**Lemma 7** (BV estimate in space [20]). Assume (8), (9), and (20) hold, and fix any $\varepsilon > 0$. Then

$$||\partial_x u_\varepsilon(t, \cdot)||_{L^1(R)} \leq |u_0|_{BV(R)} + 24t \|u_0\|_{L^2(R)}^2, \quad t \geq 0. \tag{21}$$

**Lemma 8** ($L^\infty$-estimate [20]). Assume (8), (9), and (20) hold, and fix any $\varepsilon > 0$. Then

$$||u_\varepsilon(t, \cdot)||_{L^\infty(R)} \leq |u_0|_{BV(R)} + 24t \|u_0\|_{L^2(R)}^2, \quad t \geq 0. \tag{22}$$

**Lemma 9** (BV estimate in time). Assume (8), (9), and (20) hold, and fix any $\varepsilon > 0$. Then

$$||\partial_t u_\varepsilon(t, \cdot)||_{L^1(R)} \leq C_t, \quad t \geq 0, \tag{23}$$

where the constant $C_t := (|u_0|_{BV(R)} + 24t \|u_0\|_{L^2(R)}^2 + |\gamma| (|u_0|_{BV(R)} + 24t \|u_0\|_{L^2(R)}^2) + 12 \|u_0\|_{L^2(R)}^2$ is independent of $\varepsilon$ but dependent on $t$.

**Proof.** We have, by (15), (21), and (22),

$$||\partial_t u_\varepsilon(t, \cdot)||_{L^1(R)} \leq \int_R |u_\varepsilon \partial_x u_\varepsilon| dx + \int_R |\partial_x \varepsilon P_\varepsilon| dx$$

$$\leq (||u_\varepsilon(t, \cdot)||_{L^\infty(R)} + |\gamma|) ||u_\varepsilon(t, \cdot)||_{BV(R)} + ||\partial_x P_\varepsilon(t, \cdot)||_{L^1(R)} \leq C_t. \quad \blacksquare$$

**Lemma 10** Assume (8), (9), and (20) hold, and fix any $\varepsilon > 0$. Then

$$||\partial^2_{xx} P_\varepsilon(t, \cdot)||_{L^1(R)} \leq 6 \|u_0\|_{L^2(R)}^2 + \frac{3}{2} (|u_0|_{BV(R)} + 24t \|u_0\|_{L^2(R)}^2)^2,$$

for any $t \geq 0$.

**Proof.** This is a consequence of the second equation in (7), (16) and (22). \hfill \blacksquare

3 Existence in $L^2 \cap L^4$

In this section we prove that there exists at least one weak solution to (4). Under assumption,

$$u_0 \in L^2(R) \cap L^4(R). \tag{24}$$

**Definition 11** (Weak solution). We call a function $u : R_+ \times R \to R$ a weak solution of the Cauchy problem (4) provided

i) $u \in L^\infty(R_+; L^2(R)),$

and

ii) $\partial_t u + \partial_x \left( \frac{u^2}{2} + \gamma u \right) + \partial_x P^u = 0,$  in $D'(\{0, \infty\} \times R),$  that is, $\forall \phi \in C_c^\infty([0, \infty) \times R)$ there holds the equation

$$\int_{R_+} \int_R (u \partial_t \phi + \left( \frac{u^2}{2} + \gamma u \right) \partial_x \phi - \partial_x P^u \phi) dx dt + \int_R u_0(x) \phi(0, x) dx = 0,$$

where $P^u(t, x) = G_1 \ast \left( \frac{3}{2} u^2 \right)(t, x) = \frac{3}{4} \int_R e^{-|x-y|} (u(t, y))^2 dy$.  

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Definition 12 (Entropy weak solution). We call a function $u : R^+ \times R \rightarrow R$ an entropy weak solution of the Cauchy problem (4) provided

$i)$ $u$ is a weak solution in the sense of Definition 11

$ii)$ $u \in L^\infty (0, T; BV (R))$ for any $T > 0$, and

$iii)$ for any convex $C^2$ entropy $\eta : R \rightarrow R$ with corresponding entropy flux $q : R \rightarrow R$ defined by $q' (u) = u \eta' (u)$ there holds

$$\partial_t \eta (u) + \partial_x (q(u) + \gamma \eta (u)) + \eta' (u) \partial_x P^{u} \leq 0,$$

that is, $\forall \phi \in C^\infty_c ([0, \infty) \times R), \phi \geq 0,$

$$\int_{R^+} \int_{R} (\eta(u) \partial_t \phi + (q(u) + \gamma \eta (u)) \partial_x \phi - \eta' (u) \partial_x P^{u} \phi) dx dt + \int_{R} \eta(u_0(x)) \phi(0, x) dx \geq 0. \quad (25)$$

Our main existence result is the following theorem:

Theorem 13 (Existence). Suppose (24) holds. Then there exists a function

$$u \in L^\infty (R^+; L^2 (R)) \cap L^\infty (0, T; L^4 (R)), \forall T > 0,$$

which solves the Cauchy problem (4) in $D' ([0, T) \times R)$. And the solution $u$ satisfies the following estimates for any $t \in (0, T)$ :

$$\| u(t, \cdot) \|_{L^1 (R)} \leq \| u_0 \|_{L^1 (R)} + 12 t \| u_0 \|_{L^2 (R)} ^2, \quad (26)$$

$$\| u(t, \cdot) \|_{BV (R)}, \| u(t, \cdot) \|_{L^\infty (R)} \leq \| u_0 \|_{BV (R)} + 24 t \| u_0 \|_{L^2 (R)} ^2, \quad (27)$$

$$\| u(t, \cdot) \|_{L^4 (R)} ^4 \leq e^{12 \| u_0 \|_{L^2 (R)} ^2 t} \| u_0 \|_{L^4 (R)} ^2 + 8 \| u_0 \|_{L^2 (R)} ^2 (e^{12 \| u_0 \|_{L^2 (R)} ^2 t} - 1). \quad (28)$$

Furthermore,

$$\| u(t_2, \cdot) - u(t_1, \cdot) \|_{L^1 (R)} \leq C_T |t_2 - t_1|, \forall t_1, t_2 \in [0, T], \quad (29)$$

where $C_T := (\| u_0 \|_{L^1 (R)} + 12 T \| u_0 \|_{L^2 (R)} ^2 + |\gamma| (\| u_0 \|_{L^1 (R)} + 12 T \| u_0 \|_{L^2 (R)} ^2) + 12 \| u_0 \|_{L^2 (R)} ^2$.

To avoid strict convexity of the flux function, we will use a refinement of Schonbek’s method found in [22], which we recall next.

Lemma 14 Let $\Omega$ be a bounded open subset of $R^+ \times R$. Let $f \in C^2 (R)$ satisfy

$$| f(u) | \leq C |u|^{s+1} \text{ for } u \in R, \quad |f'(u)| \leq C |u|^s \text{ for } u \in R,$$

for some $s \geq 0$.

Define functions as follows:

$$\{ I_l \in C^2 (R), \| I_l (u) \| \leq |u| (u \in R), \| I_l' (u) \| \leq 2, (u \in R), \| I_l (u) \| \leq |u| (|u| \leq l), I_l (u) = 0, (|u| \geq 2l). \}

And $f_l (u) = \int_{0}^{u} I_l' (\xi) f'(\xi) d\xi, F_l (u) = \int_{0}^{u} f_l' (\xi) f'(\xi) d\xi$.

Suppose $\{ u_n \}_{n=1}^{\infty} \subset L^{2(s+1)} (\Omega), \text{ such that the two sequences} \quad \{ \partial_t I_l (u_n) + \partial_x f_l (u_n) \}_{n=1}^{\infty}, \quad \{ \partial_t f_l (u_n) + \partial_x F_l (u_n) \}_{n=1}^{\infty}.$

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of distributions belong to a compact subset of $H^{-1}_{loc}(\Omega)$, for each fixed $l > 0$.

Then there exists a subsequence of $\{u_{n}\}_{n=1}^{\infty}$ that converges to a limit function $u \in L^{2(s+1)}(\Omega)$ strongly in $L^{r}(\Omega)$ for any $(1 \leq r \leq 2(s+1))$.

The following lemma of Murat [23] is useful:

**Lemma 15** Let $\Omega$ be a bounded open subset of $R^{N}$, $N \geq 2$. Suppose the sequence $\{\ell_{n}\}_{n=1}^{\infty}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that $\ell_{n} = \ell_{n}^{1} + \ell_{n}^{2}$, where $\{\ell_{n}^{1}\}_{n=1}^{\infty}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$ and $\{\ell_{n}^{2}\}_{n=1}^{\infty}$ lies in a bounded subset of $M_{loc}(\Omega)$. Then $\{\ell_{n}\}_{n=1}^{\infty}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$.

We now turn to the proof of Theorem 13, which will be accomplished through a series of lemmas.

**Lemma 16** ([20]) Suppose (24) holds. Then there exists a subsequence $\{u_{\varepsilon_{k}}\}_{k=1}^{\infty}$ of $\{u_{\varepsilon}\}_{\varepsilon > 0}$ and a limit function.

$$u \in L^{\infty}(\Omega) \cap L^{\infty}(0, T; L^{4}(R)) \forall T > 0,$$

such that

$$u_{\varepsilon_{k}} \rightarrow u \text{ in } L^{p}((0, T) \times R) \forall T > 0, \forall p \in [2, 4) \quad (30)$$

If, in addition, $u_{0} \in L^{1}(R)$, then $u_{\varepsilon_{k}} \rightarrow u$ in $L^{p}((0, T) \times R), \forall T > 0, \forall p \in [1, 4)$.

**Lemma 17** ([20]) Suppose (24) holds. Then

$$P_{\varepsilon_{k}} \rightarrow P^{u} \text{ in } L^{p'}((0, T) \times R) \forall T > 0, \forall p \in [1, 2). \quad (31)$$

where the sequence $\{\varepsilon_{k}\}_{k=1}^{\infty}$ and the function $u$ are constructed in Lemma 16

**Lemma 18** Suppose (24) holds. Then the limit $u$ from Lemma 16 is a weak solution of (4). Moreover, $u \in L^{\infty}(0, T; L^{4}(R))$ for each $T > 0$.

**Proof.** This is an immediate consequence of Lemma 16 and Lemma 17. ■

**Lemma 19** Suppose (24) holds. Then the weak solution $u$ from Lemma 18 satisfies the entropy inequality (25) for any convex $C^2$ entropy $\eta : R \rightarrow R$ with bounded and corresponding entropy flux $q : R \rightarrow R$ defined by $q(u) = u\eta'(u)$.

**Proof.** Let $(\eta, q)$ be as in the lemma.

$$\partial_{t}\eta(u_{\varepsilon_{k}}) + \partial_{x}(q(u_{\varepsilon_{k}}) + \gamma \eta(u_{\varepsilon_{k}})) + \eta'(u_{\varepsilon_{k}})\partial_{x} P_{\varepsilon_{k}} \leq \varepsilon k \partial_{xx}^{2} \eta(u_{\varepsilon_{k}}) \text{ in } D^{'}([0, \infty) \times R). \quad (32)$$

Observing that

$$|\eta(u)| = O(1 + u^{2}), |\eta'(u)| = O(1 + u), |q(u)| = O(1 + u^{3}).$$

we can use (30) and (31) when sending $k \rightarrow \infty$ in (32). The result is

$$\partial_{t}\eta(u) + \partial_{x}(q(u) + \gamma \eta(u)) + \eta'(u)\partial_{x} P^{u} \leq 0 \text{ in } D^{'}([0, \infty) \times R).$$

which concludes the proof of the lemma. ■

**Proof of Theorem** 13. This follows from Lemmas 18 and 19. The priori estimates in Section 2 imply immediately that the limit function $u$ satisfies (26)-(29).
Well-posedness in $L^1(R) \cap BV(R)$

In this section we prove existence of entropy weak solutions to (4) under the assumption:

$$u_0 \in L^1(R) \cap BV(R). \quad (33)$$

Theorem 20 (Existence). Suppose (33) holds. Then there exists at least one entropy weak solution to (4).

Proof. We assume then that the approximating sequence $\{u_0, \varepsilon\}_{\varepsilon > 0}$ is chosen such that (8), (9), (18) and (20) hold. Then, in view of the a priori estimates obtained in Section 2, it takes a standard argument to see that there exists a sequence of strictly positive numbers $\{\varepsilon_k\}_{k=1}^\infty$ tending to zero such that as $k \to \infty$,

$$u_{\varepsilon_k} \to u, \text{ a.e. in } R_+ \times R, \quad (34)$$

and hence $u_{\varepsilon_k} \to u$ in $L^p_{loc}(R_+ \times R)$ for all $p \in [1, \infty)$.

Thanks to (34) and estimates (19), (22) there also holds

$$u_{\varepsilon_k} \to u \quad \text{in } L^p((0, T) \times R), \forall T > 0, \forall p \in [1, \infty).$$

Let us now prove that as $k \to \infty$

$$P_{\varepsilon_k} \to P^u, \partial_x P_{\varepsilon_k} \to \partial_x P^u \quad \text{in } L^p((0, T) \times R), \forall T > 0, \forall p \in [1, \infty).$$

Using this fact and arguing as in the proof of Lemma 17 we find that

$$\|P_{\varepsilon_k} - P^u\|_{L^p((0, T) \times R)} \to 0, \quad \partial_x P_{\varepsilon_k} - \partial_x P^u\|_{L^p((0, T) \times R)} \to 0, \quad k \to \infty.$$

Following the same argument in [20], we complete the proof. ■

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