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Analysis of a Hybrid Finite Difference Scheme for the Black-Scholes Equation Governing Option Pricing

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Abstract. In this paper we present a hybrid finite difference scheme on a piecewise uniform mesh for a class of Black-Scholes equations governing option pricing which is path-dependent. In spatial discretization a hybrid finite difference scheme combining a central difference method with an upwind difference method on a piecewise uniform mesh is used. For the time discretization, we use an implicit difference method on a uniform mesh. Applying the discrete maximum principle and barrier function technique we prove that our scheme is second-order convergent in space for the arbitrary volatility and the arbitrary asset price. Numerical results support the theoretical results.

Keywords: Black-Scholes equation, option valuation, central difference scheme, upwind difference scheme, piecewise uniform mesh

1 Introduction

An option is a financial contract that gives its owner the right to buy or sell a specified amount of a particular asset at a fixed price, called the exercise price, on or before a specified date, called the maturity date. Options that can be exercised at any time up to the maturity are called American, while options that can only be exercised on the maturity data are European. Options which provide the right to buy the underlying asset are known as calls, whereas options conferring the right to sell the underlying asset are referred to as puts. It was shown by Black and Scholes [1] that these option prices satisfy a second-order partial differential equation with respect to the time horizon t and the underlying asset price x. This equation is now known as the Black-Scholes equation, and can be solved exactly when the coefficients are constants or space-independent. However, in many practical situations, numerical solutions are normally sought. Therefore, efficient and accurate numerical algorithms are essential for solving this problem accurately. The first accurate numerical approach to the Black-Schoels equations was the lattice technique proposed in Cox et al. [4] and improved in Hull and White [5]. That approach is equivalent to an explicit time-stepping scheme. Other numerical schemes based on classical finite difference methods applied to constant-coefficient heat equations have also been developed (cf. Rogers [10]; Schwartz [11]; Courtadon [3]; Wilmott et al. [15]; Cai et al. [2]; Li et al. [7]-[9]). The reason for this is that when the coefficients of the Black-Scholes equation are constant or space-independent, the equation can be transformed into a diffusion equation. In this case the problem is said to be path-independent. However, when a problem is path-dependent, this transformation is impossible, and thus the Black-Scholes equation in the original form need to be solved.

The standard finite difference method is widely applied to valuating the option pricing problems, see Seydel [12], Tavella and Randall [13] and Wilmott et al. [15]. However, as stated in Seydel [12], the standard finite difference method encounters some disadvantages. It is well known that when using the standard finite difference method to solve those problems involving the convection-diffusion operator, such as the Black-Scholes partial differential operator, numerical difficulty can be caused. The main reason is that when the volatility or the asset price is small, the Black-Scholes partial differential operator becomes a

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convection-dominated operator. Hence, the implicit Euler scheme with central spatial difference method will lead to nonphysical oscillations in the computed solution. This is due to a loss in stability. The implicit Euler scheme with upwinded spatial difference method do not have this disadvantage, but this difference scheme is only first-order convergent. Wang [14] applied a fitted finite volume scheme to solve the Black-Scholes equation, and showed that the fitted volume scheme is also first-order convergent.

In this paper we present a stability hybrid difference scheme on a piecewise uniform mesh. Our hybrid difference scheme used central difference method whenever the local mesh size is small enough to ensure the stability of that scheme. Otherwise we use an upwind difference scheme. Our scheme is stable for the arbitrary volatility and the arbitrary asset price. Applying the discrete maximum principle and barrier function technique we prove that our scheme is second-order convergent in space for the arbitrary volatility and the arbitrary asset price. Without loss of generality, we shall discuss the method using the model for European options in our paper. Naturally, the method is applicable to American option if it is used together with a technique for free boundary problems.

The rest of the paper is organized as follows. In the next section we discuss the continuous model of the Black-Scholes equations. The discretization method is described in section 3. In section 4, we present a stability and error analysis for the hybrid finite difference scheme. It is shown that the finite difference solution converges to the exact solution at the rate of $O(h^2+\tau)$. Finally, numerical experiments are provided to support these theoretical results in section 5.

Notation. Throughout the paper, C will denote a generic positive constant (possibly subscripted) that is independent of the mesh. Note that C is not necessarily the same at each occurrence.

2 The continuous problem

Let V denote the value of a European call or put option and let x denote the price of the underlying asset. It is well known that V satisfies the following Black-Scholes equation (see, for example, Wilmott et al. [15]):

$$-\frac{\partial V}{\partial t}-\frac{1}{2}\sigma^2(t)x^2\frac{\partial^2 V}{\partial x^2}-(r(t)x-D(x,t))\frac{\partial V}{\partial x}+rV=0, \quad \text{for } (x,t)\in\Omega \tag{2.1}$$

with compatibility boundary and final (or payoff) conditions

$$V(0,t)=g_1(t), \quad V(X,t)=g_2(t), \quad t\in[0,T), \tag{2.2}$$

$$V(x,T)=g_3(x), \quad x\in\bar{I}, \tag{2.3}$$

where $\Omega=I\times(0,T), I=(0,X)\subset R$, $\sigma(t)>0$ denotes the volatility of the asset, $T>0$ the expiry date, $r(t)\geq 0$ the interest rate and $D(x,t)$ the dividend.

Wang [14] transform (2.1) with the non-homogeneous Dirichlet boundary conditions in (2.2) and (2.3) into the following self-adjoint form with the homogeneous boundary condition:

$$Lu\equiv-\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}[a(t)x^2\frac{\partial u}{\partial x}+b(x,t)xu]+c(x,t)u=f(x,t), \text{ for } (x,t)\in\Omega, \tag{2.4}$$

$$u(0,t)=u(X,t)=0, \quad \text{for } t\in[0,T), \qquad u(x,T)=g(x,T), \quad \text{for } x\in\bar{I}, \tag{2.5}$$

where $a(t)=\frac{1}{2}\sigma^2(t), b(x,t)=r(t)-\frac{D(x,t)}{x}-\sigma^2, c(x,t)=2r(t)-\sigma^2-\frac{\partial D}{\partial x}$ and $f(x,t)$ are sufficiently smooth functions. We assume that $a(t)\geq\alpha>0, \beta^*\geq b(x,t)\geq\beta>0$. For the sake of simplicity we shall also assume that $c(x,t)-b(x,t)-x\frac{\partial b}{\partial x}\geq 0$. This can always be achieved by a transformation $u=\tilde{u}\exp(\chi x)$, with χ chosen appropriately. We also assume that the problem satisfies sufficient regularity and compatibility conditions which guarantee the problem has a unique solution $u(x,t)\in C^2(\bar{\Omega})\cup C^4(\Omega)$. Our interest lies in constructing higher order numerical method for the Black-Scholes equation.

3 Discretization

In this section we will describe a piecewise-uniform mesh and a hybrid finite difference scheme.

The use of central difference scheme on uniform mesh may produces nonphysical oscillations in the computed solution. To overcome this oscillation we use a piecewise uniform mesh Ω^N on the space interval $[0, X]$:

$$x_i = \begin{cases} h & i = 1, \\ h[1 + \frac{\alpha}{\beta^*}(i-1)] & i = 2, \cdots, N, \end{cases}$$

where

$$h = \frac{X}{1 + \frac{\alpha}{\beta^*}(N-1)}.$$

For the time discretization, we use a uniform mesh Ω^K on $[0, T]$ with K mesh elements. Then the piecewise uniform mesh $\Omega^{N,K}$ on Ω is defined to be the tensor product $\Omega^{N,K} = \Omega^N \times \Omega^K$. It is easy to see that the mesh sizes $h_i = x_i - x_{i-1}$ and $\tau_j = t_j - t_{j-1}$ satisfy

$$h_i = \begin{cases} h & \text{for } \ i = 1, \\ \frac{\alpha}{\beta^*}h & \text{for } \ i = 2, \cdots, N, \end{cases}$$

and $\tau = \tau_j = T/K \quad \text{for } \ j = 1, \cdots, K$ respectively.

We discretize (2.4) using a central difference scheme on the uniform mesh $[x_2, x_N]$. Integrating both sides of (2.4) over $(x_{i-1/2}, x_{i+1/2})$ we have

$$-\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial u}{\partial t}\mathrm{d}x - [a(t)x^2\frac{\partial u}{\partial x} + b(x,t)xu]_{x_{i-1/2}}^{x_{i+1/2}} + \int_{x_{i-1/2}}^{x_{i+1/2}} c(x,t)u\mathrm{d}x$$
$$= \int_{x_{i-1/2}}^{x_{i+1/2}} f(x,t)\mathrm{d}x$$

for $i = 2, \cdots, N-1$. Applying the mid-point quadrature rule and difference discretization we obtain from the above

$$L_v^{N,K}U_i^j \equiv -\frac{U_i^{j+1} - U_i^j}{\tau_{j+1}} - \frac{1}{\hbar_i}(a^j x_{i+1/2}^2 \frac{U_{i+1}^j - U_i^j}{h_{i+1}} - a^j x_{i-1/2}^2 \frac{U_i^j - U_{i-1}^j}{h_i})$$
$$-\frac{1}{\hbar_i}(b_{i+1/2}^j x_{i+1/2}\frac{U_i^j + U_{i+1}^j}{2} - b_{i-1/2}^j x_{i-1/2}\frac{U_{i-1}^j + U_i^j}{2}) + c_i^j U_i^j = f_i^j \tag{3.1}$$

for $i = 2, \cdots, N-1$, where $\hbar_i = (h_i + h_{i+1})/2$, $s_{i-1/2} = (s_{i-1} + s_i)/2$ for any function $s(x)$.

To get a stable second-order discretisation we use the following upwind difference scheme at x_1:

$$L_u^{N,K}U_i^j \equiv -\frac{U_i^{j+1} - U_i^j}{\tau_{j+1}} - a^j x_i^2 \frac{1}{\hbar_i}(\frac{U_{i+1}^j - U_i^j}{h_{i+1}} - \frac{U_i^j - U_{i-1}^j}{h_i}) - 2a^j x_i \frac{U_{i+1}^j - U_i^j}{h_{i+1}}$$
$$-\frac{b_{i+1}^j x_{i+1} U_{i+1}^j - b_i^j x_i U_i^j}{h_{i+1}} + c_i^j U_i^j = f_i^j \qquad \text{for } \ i = 1. \tag{3.2}$$

We set

$$L_h^{N,K}U_i^j = \begin{cases} L_u^{N,K}U_i^j & \text{for } \ i = 1, \\ L_v^{N,K}U_i^j & \text{for } \ i = 2, \cdots, N. \end{cases} \tag{3.3}$$

Then our scheme reads: Find $U_i^j \in R^{N+1} \times R^{K+1}$ with

$$L_h^{N,K}U_i^j = f_i^j \qquad \text{for } \ i = 1, 2, \cdots, N-1, \ j = K-1, \cdots, 1, 0, \tag{3.4}$$
$$U_0^j = U_N^j = 0 \qquad \text{for } \ j = K, \cdots, 1, 0, \tag{3.5}$$
$$U_i^K = g_i^K \qquad \text{for } \ i = 1, \cdots, N-1. \tag{3.6}$$

4 Analysis of the method

Our analysis is based on discrete maximum principle, truncation error analysis and barrier function techniques.

Lemma 1 *(Maximum principle) The operator $L_h^{N,K}$ defined by (3.3) on the piecewise uniform mesh $\Omega^{N,K}$ satisfies a discrete maximum principle, i.e. if $\{v_i^j\}$ and $\{w_i^j\}$ are mesh functions that satisfy $v_0^j \leq w_0^j$, $v_N^j \leq w_N^j$ $(j = 0, 1, \cdots, K)$, $v_i^K \leq w_i^K$ $(i = 0, 1, \cdots, N)$ and $L_h^{N,K} v_i^j \leq L_h^{N,K} w_i^j$ $(i = 1, \cdots, N-1, j = K-1, \cdots, 1, 0)$, then $v_i^j \leq w_i^j$ for all i, j.*

Proof. By the assumptions of $a(t) \geq \alpha > 0, \beta^* \geq b(x,t) \geq \beta > 0$ and $c(x,t) - b(x,t) - x\frac{\partial b}{\partial x} \geq 0$, it is easy to verify that the matrix associated with $L_h^{N,K}$ is an M-matrix, as in the proof of [6, Lemma 3.1]. ■

The next lemma gives us a useful formula for the truncation error.

Lemma 2 *Let $s(x,t)$ be a smooth function defined on $\Omega^{N,K}$. Then the following estimate for the truncation error hold true:*

$$\begin{aligned} |L_h^{N,K} s_i^j - (Ls)_i^j| &\leq C\int_{t_{j-1}}^{t_{j+1}} |\frac{\partial^2 s}{\partial t^2}(x_i,t)|\mathrm{d}t + Ch\int_{x_{i-1}}^{x_{i+1}} [x_i^2|\frac{\partial^4 s}{\partial x^4}(x,t_j)| \\ &\quad + x_i|\frac{\partial^3 s}{\partial x^3}(x,t_j)|]\mathrm{d}x \qquad \text{for} \quad 2 \leq i < N, \; 0 < j < K, \end{aligned}$$

and

$$\begin{aligned} |L_h^{N,K} s_i^j - (Ls)_i^j| &\leq C\int_{t_{j-1}}^{t_{j+1}} |\frac{\partial^2 s}{\partial t^2}(x_i,t)|\mathrm{d}t + C\int_{x_{i-1}}^{x_{i+1}} [x_i^2|\frac{\partial^3 s}{\partial x^3}(x,t_j)| \\ &\quad + x_i|\frac{\partial^2 s}{\partial x^2}(x,t_j)|]\mathrm{d}x \qquad \text{for} \quad i = 1, \; 0 < j < K. \end{aligned}$$

Proof. It can be easily obtained by using Taylor's formula with the integral form of the remainder. ■

Now we can get the main result for our difference scheme.

Theorem 1 *Let u be the solution of (2.4)-(2.5) and U be the solution of the finite difference scheme (3.4)-(3.6). Then*

$$|u(x_i,t_j) - U_i^j| \leq C(h^2 + \tau) \quad \text{for} \;\; i = 0, 1, \cdots, N, \; j = K, \cdots, 1, 0.$$

Proof. From equation (2.4) we can easily get

$$|\frac{\partial u}{\partial t}| \leq C \quad \text{for} \;\; (x,t) \in \bar{\Omega} \tag{4.1}$$

by using the assumption $u \in C^2(\bar{\Omega})$ which implies that $\frac{\partial^2 u}{\partial x^2}$ is bounded.

Applying the definition of higher order derivative we have

$$|\frac{\partial^k u}{\partial x^k}| \leq Cx^{2-k} \quad \text{for} \;\; k = 3, 4, \;\; (x,t) \in \bar{\Omega}, \tag{4.2}$$

where we also have used the assumption $u \in C^2(\bar{\Omega})$. Now applying Lemma 2 we have

$$\begin{aligned} |L_h^{N,K}(u_i^j - U_i^j)| &= |L_h^{N,K} u_i^j - (Lu)_i^j| \leq C\int_{t_{j-1}}^{t_{j+1}} |\frac{\partial^2 u}{\partial t^2}(x_i,t)|\mathrm{d}t \\ &\quad + Ch\int_{x_{i-1}}^{x_{i+1}} (x_i^2|\frac{\partial^4 u}{\partial x^4}(x,t_j)| + x_i|\frac{\partial^3 u}{\partial x^3}(x,t_j)|)\mathrm{d}x \\ &\leq C(\tau + h^2), \qquad \text{for} \quad 2 \leq i < N, \; 0 < j < K, \end{aligned}$$

Table 1: Numerical results for Example 1

K	*N*	*error*	*rate*
1024	8	5.8048e-2	1.349
	16	2.2790e-2	0.779
	32	1.3281e-2	0.919
	64	7.0249e-3	0.966
	128	1.8175e-3	-

Table 2: Numerical results for Example 2

K	*N*	*error*	*rate*
1024	8	6.0301e-2	1.308
	16	2.4357e-2	1.575
	32	8.1775e-3	1.754
	64	2.4239e-3	1.863
	128	6.6643e-4	-

and

$$\begin{aligned} |L_h^{N,K}(u_i^j - U_i^j)| &= |L_h^{N,K}u_i^j - (Lu)_i^j| \le C\int_{t_{j-1}}^{t_{j+1}} |\frac{\partial^2 u}{\partial t^2}(x_i,t)|\mathrm{d}t \\ &\quad + C\int_{x_{i-1}}^{x_{i+1}} (x_i^2|\frac{\partial^3 u}{\partial x^3}(x,t_j)| + x_i|\frac{\partial^2 u}{\partial x^2}(x,t_j)|)\mathrm{d}x \\ &\le C(\tau + h^2), \qquad \text{for} \quad i=1,\ 0<j<K, \end{aligned}$$

where we have used the estimates (4.1) and (4.2). Hence, using the barrier function $w_i^j = C(\tau + h^2)(1 + T - t_j)$ (with the constant C sufficient large), Lemma 1 implies that for all i, j, $|u(x_i,t_j) - U_i^j| \le C(h^2 + \tau)$ which completes the proof. ■

5 Numerical experiments

In this section we verify experimentally the theoretical results obtained in the preceding section. Errors and convergence rates for the hybrid difference scheme are presented for two test problems.

Example 1. Consider the problem

$$\begin{aligned} &-\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}[2x^2\frac{\partial u}{\partial x} + xu] + 2u = f(x,t), \ \text{ for } \ (x,t) \in (0,1)\times(0,1), \\ &u(0,t) = u(1,t) = 0, \quad \text{for} \ \ t \in [0,1), \\ &u(x,1) = (1-x^3)(e^x - 1) + ex(1-x), \quad \text{for} \ \ x \in [0,1], \end{aligned}$$

where $f(x,t)$ is chosen such that $u(x,t) = (1-x^3)(e^x-1) + e^t x(1-x)$.

Example 2. Consider the problem

$$\begin{aligned} &-\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}[(\frac{1}{2} + t)x^2\frac{\partial u}{\partial x} + (1+xt)xu] + 3e^t u = f(x,t), \ \text{ for } \ (x,t) \in (0,1)\times(0,1), \\ &u(0,t) = u(1,t) = 0, \quad \text{for} \ \ t \in [0,1), \\ &u(x,1) = (1-x^3)(e^x - 1) + x(1-x), \quad \text{for} \ \ x \in [0,1], \end{aligned}$$

where $f(x,t)$ is chosen such that $u(x,t) = t(1-x^3)(e^x-1) + t^3 x(1-x)$.

For our tests we take $K = 1024$ which is a sufficiently large choice to bring out second-order convergence in space. We measure the accuracy in the discrete maximum norm $\|u - U\|_\infty$. The rates of convergence r^N are computed using the following formula: $r^N = \log_2(\frac{\|u-U^N\|_\infty}{\|u-U^{2N}\|_\infty})$.

The Table 1 and 2 correspond to the above problems respectively. The numerical results are clear illustrations of the convergence estimate of Theorem 1. They indicate that the theoretical results are fairly sharp.

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