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Solution of the Coupled Klein-Gordon Schrödinger Equation Using the Modified Decomposition Method

Santanu Saha Ray *
Heritage Institute of Technology
Anandapur, P.O.-East Kolkata Township
Kolkata-700107, India

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Abstract: The modified decomposition method has been implemented for solving a coupled Klein-Gordon Schrödinger equation. We consider a system of coupled Klein-Gordon Schrödinger equation with appropriate initial values using the modified decomposition method. The method does not need linearization, weak nonlinearity assumptions or perturbation theory. The numerical solutions of coupled Klein-Gordon Schrödinger equation have been represented graphically.

Key words: Adomian decomposition method; Modified decomposition method; Coupled Klein-Gordon Schrödinger equation

1 Introduction

In this paper, we shall consider the coupled nonlinear Klein-Gordon-Schrödinger equations in the form:

$$\begin{cases} u_{tt} - c^2 u_{xx} + u + |v|^2 = 0 \\ iv_t + v_{xx} + uv = 0 \end{cases} \tag{1}$$

The modified decomposition method has been applied for solving coupled Klein-Gordon-Schrödinger equations which play an important role in modern physics.

Fan et al. [1] have been proposed an algebraic method to obtain the explicit exact solutions for coupled Klein-Gordon-Schrödinger (K-G-S) equations. Recently, The Jacobi elliptic function expansion method has been applied to obtain the solitary wave solutions for coupled K-G-S equations [2]. We used to get the explicit series solutions of the coupled K-G-S equations without use of any linearization or transformation method by the Adomian's decomposition method (in short ADM) [3, 4]. The ADM, which accurately computes the series solution, is of great interest to applied sciences [3-5]. The method provides the solution in a rapidly convergent series with components that can be elegantly computed. The nonlinear equations are solved easily and elegantly without linearizing the problem by using the ADM [3, 4].

In this paper, we use the modified decomposition method (in short MDM) to obtain the numerical solutions of the coupled sine-Gordon equations (1). Large classes of linear and nonlinear differential equations, both ordinary as well as partial, can be solved by the Adomian decomposition method [3-39]. A reliable modification of Adomian decomposition method has been done by Wazwaz [40]. The decomposition method provides an effective procedure for analytical solution of a wide and general class of dynamical systems representing real physical problems [3-8, 10, 12-18, 21-23, 26-36, 38, 39]. This method efficiently works for initial-value or boundary-value problems and for linear or nonlinear, ordinary or partial differential equations and even for stochastic systems. Moreover, we have the advantage of a single global method for solving ordinary or partial differential equations as well as many types of other equations. Recently, the

*Corresponding author. *E-mail address*: santanusaharay@yahoo.com

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solution of fractional differential equation has been obtained through Adomian decomposition method by the researchers [36-38]. The method has features in common with many other methods, but it is distinctly different on close examinations, and one should not be mislead by apparent simplicity into superficial conclusions [3, 4].

2 Analysis of the Method

We consider the coupled K-G-S equations (1) in the operator form

$$\begin{cases} L_{tt}u = c^2 L_{xx}u - u - N(u,v) \\ L_t v = iL_{xx}v + iM(u,v) \end{cases} \tag{2}$$

where $L_t \equiv \frac{\partial}{\partial t}$, $L_{tt} \equiv \frac{\partial^2}{\partial t^2}$ and $L_{xx} \equiv \frac{\partial^2}{\partial x^2}$ symbolize the linear differential operators and the notations $N(u,v) = |v|^2$ and $M(u,v) = uv$ symbolize the nonlinear operators.

Applying the two-fold integration inverse operator $L_{tt}^{-1} \equiv \int_0^t \int_0^t (\bullet) dt dt$ to the system (2) and using the specified initial conditions yields

$$\begin{cases} u(x,t) = u(x,0) + tu_t(x,0) + c^2 L_{tt}^{-1} L_{xx}u - L_{tt}^{-1}u - L_{tt}^{-1}N(u,v) \\ v(x,t) = v(x,0) + iL_t^{-1}L_{xx}v + iL_t^{-1}M(u,v) \end{cases} \tag{3}$$

The Adomain decomposition method [3, 4] assumes an infinite series solutions for unknown function $u(x,t)$ and $v(x,t)$ given by

$$\begin{cases} u(x,t) = \sum\limits_{n=0}^{\infty} u_n(x,t) \\ v(x,t) = \sum\limits_{n=0}^{\infty} v_n(x,t) \end{cases} \tag{4}$$

and nonlinear operators $N(u,v) = |v|^2$ and $M(u,v) = uv$ by the infinite series of Adomain polynomials given by

$$N(u,v) = \sum_{n=0}^{\infty} A_n(u_0, u_1, ..., u_n, v_0, v_1, ..., v_n), \quad M(u,v) = \sum_{n=0}^{\infty} B_n(u_0, u_1, ..., u_n, v_0, v_1, ..., v_n)$$

where A_n and B_n are the appropriate Adomian's polynomial which are generated according to algorithm determined in [3, 4]. For nonlinear operator $N(u,v)$, these polynomials can be defined as

$$A_n(u_0, u_1, ..., u_n, v_0, v_1, ..., v_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N\left(\sum_{k=0}^{\infty} \lambda^k u_k, \sum_{k=0}^{\infty} \lambda^k v_k \right) \right]_{\lambda=0}, \quad n \geq 0 \tag{5}$$

Similarly for nonlinear operator $M(u,v)$,

$$B_n(u_0, u_1, ..., u_n, v_0, v_1, ..., v_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[M\left(\sum_{k=0}^{\infty} \lambda^k u_k, \sum_{k=0}^{\infty} \lambda^k v_k \right) \right]_{\lambda=0}, \quad n \geq 0 \tag{6}$$

This formulae is easy to set computer code to get as many polynomial as we need in calculation of the numerical as well as explicit solutions. For the sake of convenience of the readers, we can give the first few Adomian polynomials for $N(u,v) = |v|^2$, $M(u,v) = uv$ of the nonlinearity as

$$A_0 = v_0 \overline{v}_0$$

$$A_1 = v_1 \overline{v}_0 + v_0 \overline{v}_1$$

$$A_2 = v_2 \overline{v}_0 + v_0 \overline{v}_2 + v_1 \overline{v}_1$$

....

and

$$B_0 = u_0 v_0$$

$$B_1 = u_1 v_0 + u_0 v_1$$

$$B_2 = u_2 v_0 + u_0 v_2 + u_1 v_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and so on, the rest of the polynomials can be constructed in a similar manner.

Substituting the initial conditions into (3) identifying the zeroth components u_0 and v_0, then we obtain the subsequent components by the following recursive equations by using the standard ADM

$$\begin{array}{l} u_{n+1} = c^2 L_{tt}^{-1} L_{xx} u_n - L_{tt}^{-1} u_n - L_{tt}^{-1} A_n \,, \quad n \geq 0 \\ v_{n+1} = i L_t^{-1} L_{xx} v_n + i L_t^{-1} B_n \,, \quad n \geq 0 \end{array} \tag{7}$$

Recently, Wazwaz [40] proposed that the construction of the zeroth component of the decomposition series can be defined in a slightly different way. In [40], he assumed that if the zeroth component $u_0 = g$ and the function g is possible to divide into two parts such as g_1 and g_2 , the one can formulate the recursive algorithm for u_0 and general term u_{n+1} in a form of the modified recursive scheme as follows:

$$\begin{array}{l} u_0 = g_1 \\ u_1 = g_2 + c^2 L_{tt}^{-1} L_{xx} u_0 - L_{tt}^{-1} u_0 - L_{tt}^{-1} A_0 \\ u_{n+1} = c^2 L_{tt}^{-1} L_{xx} u_n - L_{tt}^{-1} u_n - L_{tt}^{-1} A_n \,, \quad n \geq 1 \end{array} \tag{8}$$

Similarly, if the zeroth component $v_0 = g'$ and the function g' is possible to divide into two parts such as g'_1 and g'_2 , the one can formulate the recursive algorithm for v_0 and general term v_{n+1} in a form of the modified recursive scheme as follows:

$$\begin{array}{l} v_0 = g'_1 \\ v_1 = g'_2 + i L_t^{-1} L_{xx} v_0 + i L_t^{-1} B_0 \\ v_{n+1} = i L_t^{-1} L_{xx} v_n + i L_t^{-1} B_n \,, \quad n \geq 1 \end{array} \tag{9}$$

This type of modification is giving more flexibility to the ADM in order to solve complicate nonlinear differential equations. In many cases the modified decomposition scheme avoids the unnecessary computation especially in calculation of the Adomian polynomials. The computation of these polynomials will be reduced very considerably by using the MDM.

It is worth noting that the zeroth components u_0 and v_0 are defined then the remaining components u_n and v_n , $n \geq 1$ can be completely determined. As a result, the components $u_0, u_1, ...$, and $v_0, v_1, ...$, are identified and the series solutions thus entirely determined. However, in many cases the exact solution in a closed form may be obtained.

The decomposition series (4) solutions are generally converges very rapidly in real physical problems [4]. The rapidity of this convergence means that few terms are required. Convergence of this method has been rigorously established by Cherruault [41], Abbaoui and Cherruault [42, 43] and Himoun, Abbaoui and Cherruault [44]. The practical solutions will be the n -term approximations ϕ_n and ψ_n

$$\begin{array}{l} \phi_n = \sum\limits_{i=0}^{n-1} u_i(x,t) \,, \quad n \geq 1 \\ \psi_n = \sum\limits_{i=0}^{n-1} v_i(x,t) \,, \quad n \geq 1 \end{array} \tag{10}$$

with

$$\begin{array}{l} \lim\limits_{n \to \infty} \phi_n = u(x,t) \\ \lim\limits_{n \to \infty} \psi_n = v(x,t) \end{array}$$

3 Implementation of the method

We first consider the application of coupled K-G-S equations (1) with the initial conditions

$$u(x,0) = -14p^2 - 6p^2\tanh^2(px), \quad u_t(x,0) = 24kp^3\tanh(px)\sec h^2(px)$$
$$v(x,0) = \left(-\frac{7p}{2} + 6p\tanh^2(px)\right)e^{ikx} \tag{11}$$

where p and k are arbitrary constants.

Using (7) and (8) with (5) and (6) respectively and considering $c^2 = \frac{4k^2p^2-1}{p^2}$ for the coupled K-G-S equations (1) and initial conditions (11) gives

$$u_0 = 0$$
$$u_1 = u(x,0) + tu_t(x,0) + c^2L_{tt}^{-1}L_{xx}u_0 - L_{tt}^{-1}u_0 - L_{tt}^{-1}A_0$$
$$= -14p^2 - 6p^2\tanh^2(px) + 24kp^3t\tanh(px)\sec h^2(px)$$
$$u_2 = c^2L_{tt}^{-1}L_{xx}u_1 - L_{tt}^{-1}u_1 - L_{tt}^{-1}A_1$$
$$= p^2(-1+4k^2p^2)t^2\sec h^5(px)(-9\cosh(px) + 3\cosh(3px) +$$
$$4kpt(-11\sinh(px) + \sinh(3px))) - p^2t^2(-7 + 4kpt\sec h^2(px)\tanh(px) - 3\tanh^2(px))$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and

$$v_0 = 0$$
$$v_1 = v(x,0) + iL_t^{-1}L_{xx}v_0 + iL_t^{-1}B_0 = \left(-\frac{7p}{2} + 6p\tanh^2(px)\right)e^{ikx}$$
$$v_2 = iL_t^{-1}L_{xx}v_1 + iL_t^{-1}B_1$$
$$= \frac{1}{2}ie^{ikx}pt(-5k^2 + 12\sec h^2(px)(k^2 - 4p^2 + 6p^2\sec h^2(px) + 4ikp\tanh(px)))$$

and so on, in this manner the other components of the decomposition series can be easily obtained of which $u(x,t)$ and $v(x,t)$ were evaluated in a series form

$$u(x,t) = -14p^2 - 6p^2\tanh^2(px) + 24kp^3t\tanh(px)\sec h^2(px)$$
$$+p^2(-1+4k^2p^2)t^2\sec h^5(px)(-9\cosh(px) + 3\cosh(3px) +$$
$$4kpt(-11\sinh(px) + \sinh(3px))) - p^2t^2(-7 + 4kpt\sec h^2(px)\tanh(px) - 3\tanh^2(px)) + \ldots\ldots \tag{12}$$

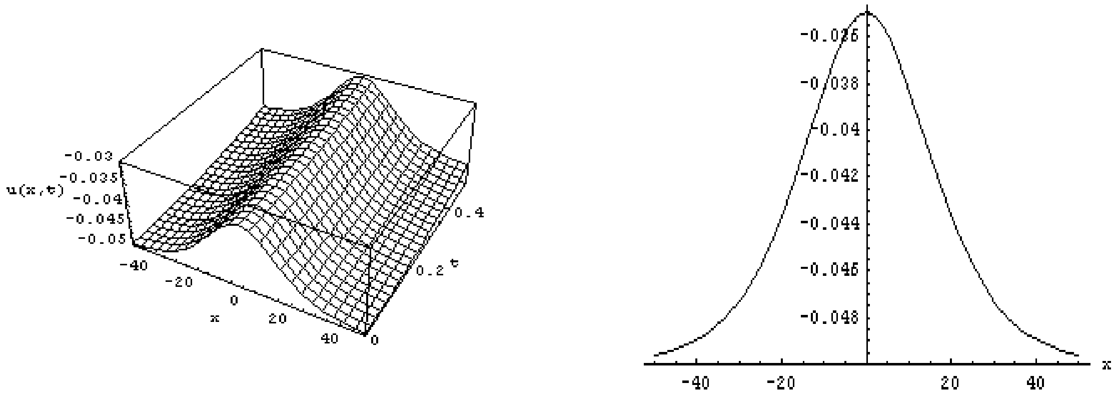
$$\begin{aligned} v(x,t) = &\left(-\frac{7p}{2} + 6p\tanh^2(px)\right)e^{ikx} \\ &+ \frac{1}{2}ie^{ikx}pt(-5k^2 + 12\sec h^2(px)(k^2 - 4p^2 + 6p^2\sec h^2(px) + 4ikp\tanh(px))) + \ldots\ldots \end{aligned} \tag{13}$$

follows immediately with the aid of *Mathematica* [45].

4 Numerical Results and Discussions

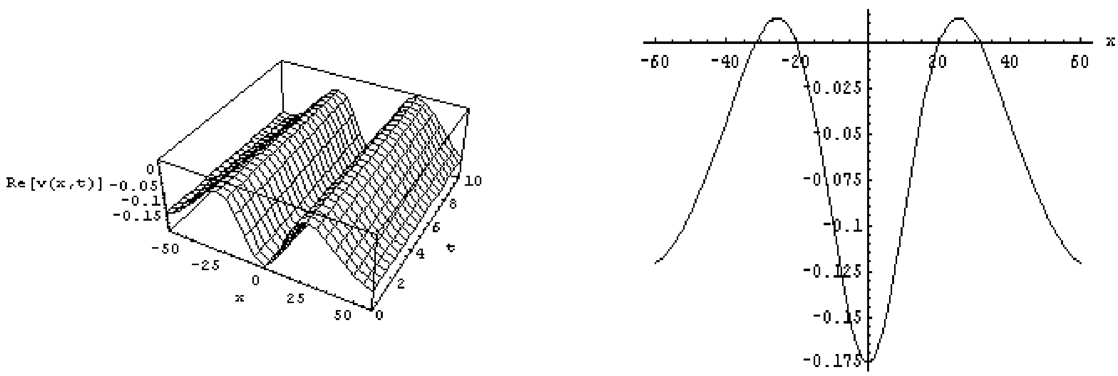
In the present numerical experiment, Eqs. (12) and (13) have been used to draw the graphs as shown in Figs.1, 2, 3 and 4 respectively.

The numerical solutions of the coupled K-G-S equations (1) have been shown in Fig. 1, Fig. 2, Fig. 3 and Fig. 4 with the help of five-term and four-term approximations ϕ_5 and ψ_4 for the decomposition series solutions of $u(x,t)$ and $v(x,t)$ respectively. In the present numerical computation we have assumed $p = 0.05$ and $k = 0.05$. Figures 1, 2, 3 and 4 have been drawn using the *Mathematica* software [45].



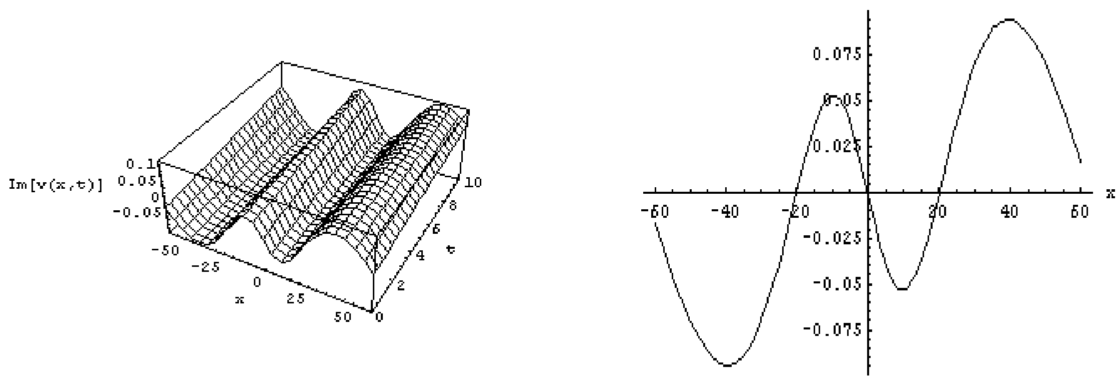
(a) (b)

Figure 1: (a)The decomposition method solution for $u(x,t)$, (b) Corresponding solution for $u(x,t)$ when $t=0$



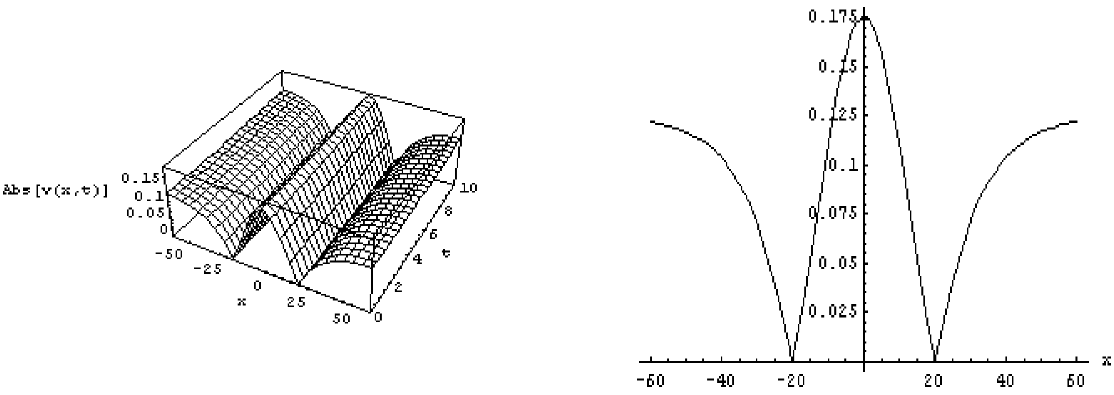
(a) (b)

Figure 2: (a) The decomposition method solution for $Re(v(x,t))$, (b) Corresponding solution for $Re(v(x,t))$ when $t=0$



(a) (b)

Figure 3: (a) The decomposition method solution for $Im(v(x,t))$, (b) Corresponding solution for $Im(v(x,t))$ when $t=0$



(a) (b)

Figure 4: (a) The decomposition method solution for $Abs(v(x,t))$, (b) Corresponding solution for $Abs(v(x,t))$ when $t = 0$

5 Conclusion

In this paper, the modified decomposition method was used for finding the solutions for the coupled K-G-S equations with initial conditions. The approximate solutions to the equations have been calculated by using the MDM without any need to a transformation techniques and linearization of the equations. Additionally, it does not need any discretization method to get numerical solutions. This method thus eliminates the difficulties and massive computation work. The decomposition method is straightforward, without restrictive assumptions and the components of the series solution can be easily computed using any mathematical symbolic package. Moreover, this method does not change the problem into a convenient one for the use of linear theory. It, therefore, provides more realistic series solutions that generally converge very rapidly in real physical problems.

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