Periodic Solutions for a Class of Non-autonomous Second Order Systems

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Abstract: In this paper, we investigate periodic solutions for a class of non-autonomous second-order systems by applying condition $\text{(C)}^\ast$, and give out some new solvability conditions of existence for nontrivial periodic solutions, the results generalize and improve some known results.

Keywords: periodic solutions; superquadratic; condition $\text{(C)}^\ast$; local linking

\section{Introduction and main results}

Solution theory is one of the most important aspect in nonlinearity, which is widely applied in many natural sciences such as chemistry, biology, mathematics, communication, and particularly in almost all branches of physics like fluid dynamics, plasma physics, field theory, optics, and condensed matter physics. In order to find some new exact solutions of nonlinear equations, a wealth of effective methods has been set up [11-15].

Consider the second order Hamiltonian systems:

\begin{equation}
\begin{cases}
\ddot{u}(t) + \nabla F(t, u(t)) = 0 & \text{a.e. } t \in [0, T] \\
u(0) - u(T) = \dot{u}(0) - \ddot{u}(T) = 0
\end{cases}
\end{equation}

Where $T > 0$ and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(H) $F(t, x)$ is measurable in $t$ for each $x \in \mathbb{R}^N$ and continuously differentiable in $x$ for a.e.$t \in [0, T]$, and there exist $a \in C(R^+, R^+), b \in L^1(0, T; R^+)$ such that

\[|F(t, x)| \leq a(|x|)b(t), |
abla F(t, x)| \leq a(|x|)b(t)\]

for all $x \in \mathbb{R}^N$ and a.e.$t \in [0, T]$.

Rabinowitz established the existence of periodic solutions for system (1) under the following superquadratic condition in [1]: there exist $\mu > 2$ and $L > 0$ such that for all $|x| \geq L$ and $t \in [0, T]$

\[0 < \mu F(t, x) \leq (\nabla F(t, x), x).
\]

Since then, the condition has been used extensively, we refer the reader to [2-3] and the references therein.

Recently, G. Fei in [4] has obtained the existence of periodic solutions for systems (1) under a new superquadratic conditions which is different from the superquadratic conditions in [3].

In [5-6], Z.-L. Tao has obtained the existence of periodic solutions for systems (1) under the superquadratic conditions of G. Fei which generalizes the Theorem 1.2 in [4].

In [7] we consider the following second-order systems:

\begin{equation}
\begin{cases}
\dddot{u}(t) + A\dot{u}(t) + \nabla F(t, u(t)) = 0 & \text{a.e. } t \in [0, T], \\
u(0) - u(T) = \dot{u}(0) - \ddot{u}(T) = 0
\end{cases}
\end{equation}

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where $A$ is antisymmetry constant matrix.

Motivated by [6-7], we will consider the problem (2) and give more general results than [4],[6].

**Theorem 1.** Suppose that $F$ satisfies conditions (H) and the following conditions:

$$
\lim_{|x| \to 0} \frac{F(t,x)}{|x|^2} < \frac{1}{2} \omega^2 \quad \text{uniformly for a.e.} \ t \in [0,T],
$$

$$
\liminf_{|x| \to \infty} \frac{F(t,x)}{|x|^2} = +\infty \quad \text{uniformly for a.e.} \ t \in [0,T],
$$

where $\omega^2$ is a constant. There exist $L > 0, r > 2$ and $\mu > r - 2$ such that

$$
F(t,x) \geq 0, \ \forall \ |x| \leq L \ \text{and} \ t \in [0,T],
$$

$$
\limsup_{|x| \to \infty} \frac{F(t,x)}{|x|^r} < \infty \quad \text{uniformly for a.e.} \ t \in [0,T],
$$

$$
\liminf_{|x| \to \infty} \left( \frac{\nabla F(t,x), x}{|x|^\mu} - 2F(t,x) \right) > 0 \quad \text{uniformly for a.e.} \ t \in [0,T].
$$

And there exist $\epsilon > 0$ such that

$$
\| A \| + \epsilon \frac{T}{\pi} < \frac{2\pi}{T}
$$

Then system (2) has at least one nontrivial periodic solution.

**Remark 1.** System (2) generalizes system (1) obviously, for if $A = 0$, then system (2) becomes system(1).

**Remark 2.** When $A = 0$ Theorem 1 generalizes Theorem 1 in [6]. Obviously, by (4) of Theorem 1 in [6], functions $F$ satisfies

$$
\lim_{|x| \to 0} \frac{F(t,x)}{|x|^2} = 0 \quad \text{uniformly for a.e.} \ t \in [0,T],
$$

but by (3),

$$
\lim_{|x| \to 0} \frac{F(t,x)}{|x|^2} < \frac{1}{2} \omega^2 \quad \text{uniformly for a.e.} \ t \in [0,T].
$$

Moreover Theorem 1 generalizes Theorem 1.2 in [4].

**Remark 3.** There are functions $F$ satisfying our Theorem 1 but not satisfying the corresponding conditions in [4-6]. For example,

$$
F(t,x) = (1 + \cos^2(\frac{2\pi t}{T})) \ |x|^2 \left( \ln(\frac{1}{3} \ |x|^4 - |x|^3 + \frac{1}{2} \ |x|^2 + \exp(\frac{\omega^2}{2})) \right)^2
$$

where $\omega = \frac{2\pi}{T}$.

## 2 The preliminary results

The Hilbert space $H^1_T$ is defined by:

$$
H^1_T = \{ u : [0,T] \to \mathbb{R}^N \mid u \text{ is absolutely continous, } u(0) = u(T) \text{ and } \dot{u} \in L^2(0,T; R^N) \}
$$

and is endowed with the norm

$$
\|u\| = \left( \int_0^T |u(t)|^2 \, dt + \int_0^T |\dot{u}(t)|^2 \, dt \right)^{\frac{1}{2}}.
$$

For $u \in H^1_T$, Let $\bar{u} = (\frac{1}{T}) \int_0^T u(t) \, dt$ and $\dot{u} = u(t) - \bar{u}$.

Then one has Sobolev’s inequality

$$
\|\dot{u}\|_{L^2} \leq \frac{T}{12} \|\ddot{u}\|_{L^2},
$$

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and Wirtinger’s inequality
\[
\int_0^T |\ddot{u}(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{u}|^2 dt.
\]  
(9)
for all \( u \in H^1_T \) (see proposition 1.3 in [8]).

Let \( X^1 = \{ u \in X : \int_0^T u dt = 0 \} \), \( X^2 = \mathbb{R}^N \). It is easy to know that \( X^1 \) is a subset of \( X \), and 
\[ X = X^1 \oplus X^2. \]

Moreover, there exists a positive constant \( c_0 \) such that
\[
\| u \|_\infty \leq c_0 \| u \| \tag{10}
\]
for all \( u \in H^1_T \), where \( \| u \|_\infty = \max_{0 \leq t \leq T} |u(t)| \) (see proposition 1.1 in [8]).

By the Sobolev embedded theorems, there exists \( \theta > 0 \), for all \( u \in X \), we have
\[
\| u \|_{L^\infty} \leq \theta \| u \|. \tag{11}
\]

**Lemma 1.** Define the corresponding functional \( \varphi \) on \( H^1_T \) by:
\[
\varphi(u) = \frac{1}{2} \int_0^T |\ddot{u}(t)|^2 dt + \frac{1}{2} \int_0^T (Au(t), \dot{u}(t)) dt - \int_0^T F(t, u(t)) dt. \tag{12}
\]
It follows from (H) that \( \varphi \) is continuously differentiable and the solutions of problem (2) correspond to the critical points of \( \varphi \). Moreover one has
\[
< \varphi'(u), v > = \int_0^T (\ddot{u}(t), \dot{v}(t)) dt - \int_0^T (A\dot{u}(t), v(t)) dt - \int_0^T (\nabla F(t, u(t)), v(t)) dt
\]
for all \( u, v \in H^1_T \).

**Proof.** It follows from (H) that \( \varphi \) is continuously differential, and its Fréchet derivative satisfies
\[
< \varphi'(u), v > = \int_0^T (\ddot{u}(t), \dot{v}(t)) dt + \frac{1}{2} \int_0^T (Au(t), \dot{v}(t)) dt + \frac{1}{2} \int_0^T (\dot{u}(t), Av(t)) dt - \int_0^T (\nabla F(t, u(t)), v(t)) dt.
\]

Let \( u \) be the critical point of \( \varphi \), that is \( \varphi'(u) = 0 \), thus for all \( v \in H^1_T \), we have \( < \varphi'(u), v > = 0 \).

As \( A \) is antisymmetry constant matrix and integrate partially, one has
\[
\int_0^T (\ddot{u}(t), Av(t)) dt = - \int_0^T (A\ddot{u}(t), v(t)) dt,
\]
\[
\int_0^T (Au(t), \dot{v}(t)) dt = - \int_0^T (A\dot{u}(t), v(t)) dt.
\]

Hence \( < \varphi'(u), v > = 0 \) is equivalent to
\[
\int_0^T (\ddot{u}(t), \dot{v}(t)) dt = \int_0^T (A\ddot{u}(t) + \nabla F(t, u(t))) dt.
\]
Thus \( \ddot{u}(t) \) has a weak derivative and \( \ddot{u}(t) = A\ddot{u}(t) + \nabla F(t, u(t)) \) a.e. on \([0, T]\). Moreover, the existence of a weak derivative for \( u \) and \( \ddot{u} \) implies that \( u(0) - u(T) = \ddot{u}(0) - \ddot{u}(T) = 0 \). \( \square \)

Let’s recall the definition of local linking in [4]: Let \( X \) be a Banach space with a direct decomposition
\[ X = X^1 \oplus X^2. \]
The function \( f \in C^1(X, R) \) has a local linking at 0, if for some \( r > 0 \),
\[
 f(u) \geq 0, \ u \in X^1, \| u \| < r, \\
 f(u) \leq 0, \ u \in X^2, \| u \| < r.
\]

**Lemma 2**([9]). Suppose that \( \varphi \in C^1(X, R) \) satisfies the following assumptions:

(\( \varphi_1 \)) \( \varphi \) has a local linking at 0 and \( X^1 \neq \{0\} \).

(\( \varphi_2 \)) \( \varphi \) satisfies \((C)^\ast \) condition.

(\( \varphi_3 \)) \( \varphi \) maps bounded sets into bounded sets.

(\( \varphi_4 \)) For every \( m \in N, \varphi(u) \rightarrow -\infty, |u| \rightarrow \infty, u \in X^1_m \bigoplus X^2. \)

Then \( \varphi \) has at least two critical point.

### 3 Proof of Theorem 1

Now, under the conditions of Theorem 1, we will prove that \( \varphi \) satisfies \((\varphi_1 - \varphi_4)\).

**Proposition 1.** Under the conditions of (3), (5), (8) of Theorem 1, \( \varphi \) satisfies \((\varphi_1)\).

**Proof.** Firstly, by (8), we have
\[
0 < \varepsilon < \frac{2^2}{T^2}, \text{ by condition (3), there exists a constant } \rho_1 > 0, \text{ for all } |x| \leq \rho_1 \text{ and a.e. } t \in [0, T], \text{ such that }
\]
\[
F(t, x) \leq \varepsilon |x|^2.
\]
Thus by (8), (9), (11) for all \( u \in X^1 \) and \( |u| \leq \frac{\rho_1}{r} \), i.e. \( u \leq \rho_1 \), one has
\[
\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \frac{1}{2} \int_0^T (Au(t), \dot{u}(t))dt - \int_0^T F(t, u(t))dt \\
\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \frac{1}{2} A \| \frac{T}{2\pi} \int_0^T |\dot{u}(t)|^2 dt - \varepsilon \int_0^T |u(t)|^2 dt \\
\geq \left( 1 - \frac{1}{2} \right) A \| \frac{T}{2\pi} - \frac{\varepsilon T^2}{4\pi^2} \right) \int_0^T |\dot{u}(t)|^2 dt \geq 0.
\]

By condition (5), there exists a constant \( \rho_2 > 0 \), for all \( u \in X^2 \) and \( |u| \leq \rho_2 \), we have
\[
\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \frac{1}{2} \int_0^T (Au(t), \dot{u}(t))dt - \int_0^T F(t, u(t))dt \\
= - \int_0^T F(t, u(t))dt \leq 0.
\]

Let \( r = \min \{\rho_1, \rho_2\} \), for \( r > 0 \), we get
\[
\varphi(u) \geq 0, \ u \in X^1, \| u \| < r, \\
\varphi(u) \leq 0, \ u \in X^2, \| u \| < r.
\]
So \( \varphi \) has a local linking at 0. \( \square \)

**Proposition 2.** Under the conditions of (6), (7), (8) of Theorem 1, \( \varphi \) satisfies \((C)^\ast \) condition, i.e., for every sequence \( \{u_{\alpha_n}\} \subset X_{\alpha_n}, \{u_{\alpha_n}\} \) has a convergent subsequence if \( \varphi(u_{\alpha_n}) \) is bounded and \((1 + |u_{\alpha_n}|) \| \varphi'(u_{\alpha_n}) \| \rightarrow 0 \) as \( n \rightarrow \infty \), where \( X_{\alpha_n} = X^1 \bigoplus X^2, X^1 = \mathbb{R}_{\alpha_n} \{e_0, e_1, \ldots, e_n\} \) and \( (e_i)_{i \geq 0} \) is a Hilbert basis for \( X^1 \).

**Proof.** Suppose \( \{u_{\alpha_n}\} \subset X_{\alpha_n}, \varphi(u_{\alpha_n}) \) is bounded and \((1 + |u_{\alpha_n}|) \| \varphi'(u_{\alpha_n}) \| \rightarrow 0 \) as \( n \rightarrow \infty \), then there exist constant \( M > 0 \) such that
\[
\varphi(u_{\alpha_n}) \leq M, (1 + |u_{\alpha_n}|) \| \varphi'(u_{\alpha_n}) \| \leq M
\]
for every \( n \in N \). On one hand, by (6), there exists constants \( c > 0 \) and \( \rho_3 > 0 \) such that
\[
F(t, x) \leq c |x|^2 \text{ for all } |x| \geq \rho_3 \text{ and a.e. } t \in [0, T].
\]
It follows from (H), we obtain that

\[ | F(t, x) | \leq \max_{s \in [0, \rho_1]} a(s)b(t) \text{ for all } | x | \leq \rho_3 \text{ and } \text{a.e. } t \in [0, T]. \]

So for all \( x \in \mathbb{R}^N \) and and a.e. \( t \in [0, T] \), one has

\[ F(t, x) \leq \max_{s \in [0, \rho_3]} a(s)b(t) + c | x |^r. \] (14)

It follows from (12-14) that

\[
\frac{1}{2} \int_0^T |\dot{u}_{\alpha_n}(t)|^2 dt = \varphi(u_{\alpha_n}) - \frac{1}{2} \int_0^T (A u_{\alpha_n}(t), u_{\alpha_n}(t)) dt + \int_0^T F(t, u_{\alpha_n}(t)) dt \\
\leq M + c_1 + \frac{1}{2} | A | T^2 \| \dot{u}_{\alpha_n} \|^2 + c \int_0^T | u_{\alpha_n} |^r dt
\]

where \( c_1 = \max_{s \in [0, \rho_3]} a(s) \int_0^T b(t) dt \).

Then by Hölder inequality we obtained

\[
\frac{1}{2} (1 - | A | T^2) \| u_{\alpha_n} \|^2 \\
\leq M + c_1 + c \int_0^T | u_{\alpha_n} |^r dt + \frac{1}{2} (1 - | A | T^2) \int_0^T | u_{\alpha_n} |^2 dt \\
\leq M + c_1 + c \int_0^T | u_{\alpha_n} |^r dt + \frac{1}{2} (1 - | A | T^2) T^{\frac{r+2}{r}} (\int_0^T | u_{\alpha_n} |^r dt)^{\frac{r}{r}}. \] (15)

On the other hand, by (7), there exist constant \( c_2 > 0 \) and \( \rho_4 > 0 \) such that

\[ (\nabla F(t, x), x) - 2F(t, x) \geq c_2 | x |^4 > 0 \]

for all \( | x | \geq \rho_4 \) and a.e. \( t \in [0, T] \).

By (H), we obtained

\[ | (\nabla F(t, x), x) - 2F(t, x) | \leq c_3 b(t) \]

for every \( | x | \leq \rho_4 \) and a.e. \( t \in [0, T] \), where \( c_3 = (2 + \rho_4) \max_{0 \leq s \leq \rho_4} a(s) \).

Hence for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \) we obtain that

\[ (\nabla F(t, x), x) - 2F(t, x) \geq c_2 | x |^4 - c_3 b(t) - c_4 | x |^r. \]

Then we have

\[
3M \geq 2 \varphi(u_{\alpha_n}) - (\varphi'(u_{\alpha_n}), u_{\alpha_n}) \\
= \int_0^T ((\nabla F(t, \alpha_n), x) - 2F(t, \alpha_n)) dt \\
\geq c_2 \int_0^T | u_{\alpha_n} |^4 dt - T c_2 \rho_4^4 - c_3 \int_0^T b(t) dt.
\]

So \( \int_0^T | u_{\alpha_n} |^r dt \) is bounded.

If \( \mu \geq r \), using Hölder inequality, standard computations shows that

\[
\int_0^T | u_{\alpha_n} |^r dt \leq T^\frac{r-\mu}{r} (\int_0^T | u_{\alpha_n} |^\mu dt)^{\frac{r}{r}}.
\]
This together with (15) yields \( |u_{\alpha_n}| \) is bounded.

If \( \mu \leq r \), by (10) we obtain

\[
\int_0^T |u_{\alpha_n}|^r \, dt = \int_0^T |u_{\alpha_n}|^{-\mu} |u_{\alpha_n}|^\mu \, dt \\
\leq \| u_{\alpha_n} \|_{\infty}^{\mu} \int_0^T |u_{\alpha_n}|^\mu \, dt \\
\leq c_0^{\mu-\mu} \| u_{\alpha_n} \|^{r-\mu} \int_0^T |u_{\alpha_n}|^\mu \, dt.
\]

Thus by (15) and \( r - \mu < 2 \), we know that \( |u_{\alpha_n}| \) is bounded too. Hence \( |u_{\alpha_n}| \) is bounded.

In a similar way to proposition 4.1 in [8], we can prove \( \{u_{\alpha_n}\} \) has a convergent subsequence. Hence \( \varphi \) satisfies condition \((C)^*\).

**Proposition 3.** Under the conditions of (4),(5),(8) of Theorem 1, \( \varphi \) satisfies \((\varphi_4)\).

**Proof.** By conditions [4-5], there exist enough large constant \( M_1 > 0, M_2 > 0 \), for all \((t, x) \in [0, T] \times \mathbb{R}^N\), we have

\[
F(t, x) \geq M_1 |x|^2 - M_2.
\]

So for all \( u \in X^1_m \ominus X^2 \), one has

\[
\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \frac{1}{2} \int_0^T (Au(t), \dot{u}(t)) \, dt - \int_0^T F(t, u(t)) \, dt \\
\leq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \frac{1}{2} \| A \| \frac{T}{2\pi} \int_0^T |\dot{u}(t)|^2 \, dt - M_1 \int_0^T |u|^{2_2} \, dt + TM_2 \\
\leq \left( \frac{1}{2} + \frac{1}{2} \| A \| \frac{T}{2\pi} \right) \| u \|^{2_2} - M_1 \| u \|^{2_2} + TM_2
\]

As we all know \( \text{dim}(X^1_m \ominus X^2) < \infty \), then there exist constants \( c_4 > 0 \) and \( c_5 > 0 \), such that

\[
c_4 \| u \| \leq \| u \|_{L^2} \leq c_5 \| u \|.
\]

Thus we have

\[
\varphi(u) \leq \left( \frac{1}{2} + \frac{1}{2} \| A \| \frac{T}{2\pi} - c_4M_1 \right) \| u \|^2 + TM_2
\]

Let \( M_1 \) is large enough, we get

\[
\varphi(u) \to \infty \quad \text{as} \quad \| u \| \to \infty.
\]

Hence \((\varphi_4)\) is proved. \(\Box\)

**Proof of Theorem 1.** According the conclusion of [10], we know, under the conditions of Theorem 1, \( \varphi \) maps bounded sets into bounded sets, thus \( \varphi \) of Lemma 2 holds. By Proposition 1, 2, 3 and Lemma 2 we know that system (2) has at least one nontrivial periodic solution. \(\Box\)

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**References**


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