Properties of Distribution Class and Spectral Class for a Self-similar Set

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Abstract: In this paper, we study the properties of distribution class and spectral class for a self-similar set. We give a relation between distribution sets in the distribution class and the subsets in a spectral class generated by the lower (upper) local dimensions of a self-similar measure. Then we compute their respective Hausdorff and packing dimensions, and compare our results with those of others. Finally, we attain that all spectral classes by the self-similar measures and their lower and upper local dimensions are classified into two classes: the lower distribution class and the upper distribution class.

Keywords: distribution class; spectral class; Hausdorff dimension; packing dimension; local dimension

1 Introduction

Many authors [1-4] have studied fractal properties of self-similar attractors. A self-similar Cantor set was investigated in [5] to study the relation between its spectral class and distribution class. In this paper, we study the corresponding properties of a self-similar set in $\mathbb{R}^d$. We relate a spectral class by the lower (upper) local dimensions of a self-similar measure with the class by the lower or upper distribution sets (see Section 2). The relationship gives the comparison of a subset in a spectral class with another subset in a different spectral class via a distribution set (see Corollary 3.4). Thus all spectral classes by the self-similar measures and their lower and upper local dimensions are classified into two classes: the lower distribution class and the upper distribution class. We compute Hausdorff and packing dimensions of the lower (upper) distribution sets to find singularities when calculating packing dimensions of some lower (upper) distribution sets (see Corollary 3.6). Using these results with the relationship, we compute the values of dimensions of the subsets composing a spectral class generated by a self-similar measure and its lower (upper) local dimensions (see Corollary 3.7). Except for a singular self-similar measure, every self-similar measure gives two spectral classes which are derived from its lower local dimensions and its upper local dimensions.

2 Definitions and notations

We denote $F$ a self-similar set (its diameter is 1) which is the attractor of the similarities $S_1, S_2, \cdots, S_N$ on a closed subset $D \subset \mathbb{R}^d$ with the ratios $r_i \in (0, 1)$ and $1 - \sum_{i=1}^{N} r_i > 0$. Suppose that the open set condition (OSC) holds for the similarities $S_i$ on $\mathbb{R}^d$ with ratios $r_i \in \{a, b\}$. Let

$$N_a = \#\{r_i = a, \ i = 1, 2, \cdots, N\}$$

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$N_b = \# \{ \sigma_i = b, \ i = 1, 2, \cdots, N \}$

then $N_a + N_b = N$. Denote

$$D_a^k = S_{i_k}(D) \text{ when } r_{i_k} = a, \ k = 1, 2, \cdots, N_a,$$

$$D_b^k = S_{j_k}(D) \text{ when } r_{j_k} = b, \ k = 1, 2, \cdots, N_b.$$

Let $D_{i_1 \cdots i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(D)$, where $i_j \in \{1, 2, \cdots, N \}$ and $1 \leq j \leq k$. We note that if $x \in F$, then there is a $\sigma \in \{1, 2, \cdots, N\}^N$ such that $\bigcap_{k=1}^{\infty} D_{\sigma[k]} = \{x\}$, here $\sigma[k] = i_1 i_2 \cdots i_k$, where $\sigma = i_1 i_2 \cdots i_k i_{k+1} \cdots$. We say $\sigma$ is the address of the point $x$. If $x \in F$ and $x \in D_\sigma$, where $\sigma \in \{1, 2, \cdots, N\}^k$, $c_k(x)$ denotes $D_\sigma$ and $|c_k(x)|$ denotes the diameter of $c_k(x)$ for each $k = 0, 1, 2, \cdots$. Let $p \in (0, \frac{1}{N_a})$ and we denote $\mu_p$ a self-similar Borel probability measure on $F$ satisfying

$$\mu_p(D_a^k) = p, \ k = 1, 2, \cdots, N_a$$

and

$$\mu_p(D_b^k) = \frac{1 - p N_a}{N_b}, \ k = 1, 2, \cdots, N_b.$$

Here $\dim(F)$ denotes the Hausdorff dimension of $F$ and $\dim(F)$ denotes the packing dimension of $F$. We note that $\dim(F) \leq \dim(F)$ for every set $F[6]$.

For $x \in F$, we define $n_x(a | k)$ the number of digit $i_j$ (where $r_{i_j} = a, 1 \leq j \leq N_a$) occurs in the first $k$ place of the address $\sigma$ of $x$, $n_x(b | k)$ the number of digit $i_j$ (where $r_{i_j} = b, 1 \leq j \leq N_b$) occurs in the first $k$ place of the address $\sigma$ of $x$. Clearly $n_x(a | k) + n_x(b | k) = k$.

For $r \in (0, \frac{1}{N_a})$, we define the lower (upper) distribution set $F(r)$ ($\mathcal{F}(r)$) as follows:

$$F(r) = \{ x \in F : \liminf_{k \to \infty} \frac{n_x(a | k)}{N_a \cdot k} = r \},$$

$$\mathcal{F}(r) = \{ x \in F : \limsup_{k \to \infty} \frac{n_x(a | k)}{N_a \cdot k} = r \}.$$

We call $\{ F(r) : 0 < r < \frac{1}{N_a} \}$ the lower distribution class and $\{ \mathcal{F}(r) : 0 < r < \frac{1}{N_a} \}$ the upper distribution class. We write $E_\alpha^{(p)}(\mathcal{E}_\alpha^{(p)})$ for the set of points at which the lower (upper) local dimension of $\mu_p$ on $F$ is exactly $\alpha$, so that

$$E_\alpha^{(p)} = \{ x : \liminf_{r \to 0} \frac{\log \mu_p(B_r(x))}{\log r} = \alpha \},$$

$$\mathcal{E}_\alpha^{(p)} = \{ x : \limsup_{r \to 0} \frac{\log \mu_p(B_r(x))}{\log r} = \alpha \}.$$

We call $\{ E_\alpha^{(p)}(\neq \emptyset) : \alpha \in \mathbb{R} \}$ the spectral class generated by the lower local dimensions of a self-similar measure $\mu_p$ and $\{ \mathcal{E}_\alpha^{(p)}(\neq \emptyset) : \alpha \in \mathbb{R} \}$ the spectral class generated by the upper local dimensions of a self-similar measure $\mu_p$. We call $\alpha$ satisfying $E_\alpha^{(p)}(\neq \emptyset)(\mathcal{E}_\alpha^{(p)}(\neq \emptyset))$ an associated lower (upper) local dimension of $\mu_p$.

3 Main results

Lemma 3.1 For the self-similar set $F$ defined in Section 2, $\dim F = \dim F = s$, where $s$ is uniquely decided by

$$\sum_{i=1}^{N} r_i^s = N_a a^s + N_b b^s = 1.$$

Proof. It is immediate from Theorem 2.7 of [6].
Lemma 3.2 Let $p, r \in (0, \frac{1}{N_a})$ and $g(r, p) = \frac{N_a r \log p + (1 - N_a) r \log \frac{1 - p N_a}{N_b}}{N_a r \log a + (1 - N_a) r \log b}$, consider a self-similar measure $\mu_p$ on $F$, then for a real number $s$ satisfying $N_a a_s + N_b b_s = 1$,

(1) for $0 < p < a^*$
$$\liminf_{k \to \infty} \frac{n_a(x|k)}{N_a \cdot k} = r \iff \liminf_{k \to \infty} \frac{\log \mu_p(c_k(x))}{\log |c_k(x)|} = g(r, p),$$

(2) for $a^* < p < \frac{1}{N_a}$
$$\liminf_{k \to \infty} \frac{n_a(x|k)}{N_a \cdot k} = r \iff \limsup_{k \to \infty} \frac{\log \mu_p(c_k(x))}{\log |c_k(x)|} = g(r, p),$$

(3) for $0 < p < a^*$
$$\limsup_{k \to \infty} \frac{n_a(x|k)}{N_a \cdot k} = r \iff \limsup_{k \to \infty} \frac{\log \mu_p(c_k(x))}{\log |c_k(x)|} = g(r, p),$$

(4) for $a^* < p < \frac{1}{N_a}$
$$\limsup_{k \to \infty} \frac{n_a(x|k)}{N_a \cdot k} = r \iff \liminf_{k \to \infty} \frac{\log \mu_p(c_k(x))}{\log |c_k(x)|} = g(r, p).$$

Proof. For (1), let function
$$f(x) = \frac{N_a x \log p + (1 - N_a) x \log \frac{1 - p N_a}{N_b}}{N_a x \log a + (1 - N_a) x \log b}.$$ 
As
$$f'(x) = \frac{N_a (\log p b - \log \frac{1 - p N_a}{N_b} \log a)}{[N_a x \log a + (1 - N_a) x \log b]^2},$$
we note that the function $f(x)$ is a strictly increasing function where $0 < x < \frac{1}{N_a}$ under the assumption as follows
$$\frac{\log p}{\log a} > \frac{\log \frac{1 - p N_a}{N_b}}{\log b} \iff 0 < p < a^*.$$

Assume that $\liminf_{k \to \infty} \frac{n_a(x|k)}{N_a \cdot k} = r$. Consider a convergent subsequence $\{\frac{n_a(x|k_n)}{N_a \cdot k_n}\}$ of $\{\frac{n_a(x|k)}{N_a \cdot k}\}$ whose limit is $t$. Clearly $t \geq r$. Then
$$\lim_{n \to \infty} \frac{\log \mu_p(c_{k_n}(x))}{\log |c_{k_n}(x)|} = \lim_{n \to \infty} \frac{n_a(x|k_n) \log p + (k_n - n_a(x|k_n)) \log \frac{1 - p N_a}{N_b}}{N_a t \log p + (1 - N_a t) \log \frac{1 - p N_a}{N_b}} = \frac{N_a t \log p + (1 - N_a t) \log \frac{1 - p N_a}{N_b}}{N_a \log a + (1 - N_a) \log b}.$$ 
Since $f(x)$ is strictly increasing,
$$\lim_{k \to \infty} \frac{\log \mu_p(c_k(x))}{\log |c_k(x)|} = \frac{N_a r \log p + (1 - N_a r) \log \frac{1 - p N_a}{N_b}}{N_a r \log a + (1 - N_a r) \log b}.$$ 
For the converse, assume that $\liminf_{k \to \infty} \frac{n_a(x|k)}{N_a \cdot k} = r'$, where $r' \neq r$. Similarly we get
$$\lim_{k \to \infty} \frac{\log \mu_p(c_k(x))}{\log |c_k(x)|} = \frac{N_a r' \log p + (1 - N_a r') \log \frac{1 - p N_a}{N_b}}{N_a r' \log a + (1 - N_a r') \log b}.$$ 
Then (1) follows since $f(x)$ is strictly increasing. The dual arguments give (2), (3) and (4).
Theorem 3.3 Let $s$ be the unique real number satisfying $N_a a^s + (N - N_a) b^s = 1$ and let $r \in (0, \frac{1}{N_a})$. Then

1. $F(r) = E^{(p)}_{\alpha}(r)$ if $0 < p < a^s$,
2. $E^{(p)}_{\alpha}(r) = \overline{E}^{(p)}_{\alpha}(r)$ if $a^s < p < \frac{1}{N_a}$,
3. $\overline{F}(r) = \overline{E}^{(p)}_{\alpha}(r)$ if $0 < p < a^s$,
4. $\overline{F}(r) = \overline{E}^{(p)}_{\alpha}(r)$ if $a^s < p < \frac{1}{N_a}$.

**Proof.** From [6,7], we know that

$$\liminf_{r \to 0} \frac{\log \mu_p(B_r(x))}{\log r} = \liminf_{k \to \infty} \frac{\log \mu_p(c_k(x))}{\log |c_k(x)|}$$

and

$$\limsup_{r \to 0} \frac{\log \mu_p(B_r(x))}{\log r} = \limsup_{k \to \infty} \frac{\log \mu_p(c_k(x))}{\log |c_k(x)|}.$$ 

It is immediate from Lemma 3.2.  

**Corollary 3.4** Let $s$ be the unique real number satisfying $N_a a^s + (N - N_a) b^s = 1$ and let $p \in (0, \frac{1}{N_a})$ and $\alpha \in \mathbb{R}$. For the solution $r_o$ of the equation $\alpha = g(r, p)$,

1. if $0 < p < a^s$ and $\alpha \in (\frac{\log p - p N_a}{\log b}, \frac{\log p - p N_a}{\log a})$, then $E^{(p)}_{\alpha} = F^{(p)}_{\alpha} = E^{(p'')}_{\alpha''} = \overline{E}^{(p'')}_{\alpha''}$ for $0 < p' < a^s$ and $\alpha' = g(r, p')$ and for $a^s < p'' < \frac{1}{N_a}$ and $\alpha'' \in \left(\frac{\log p''}{\log a}, \frac{1-pN_a}{N_b}\right)$ with $\alpha'' = g(r, p'')$,

2. if $a^s < p < \frac{1}{N_a}$ and $\alpha \in \left(\frac{\log p}{\log a}, \frac{\log p - p N_a}{\log a}\right)$, then $E^{(p)}_{\alpha} = \overline{F}(r_o) = E^{(p'')}_{\alpha''} = \overline{E}^{(p'')}_{\alpha''}$ for $a^s < p < \frac{1}{N_a}$ and $\alpha' = g(r, p')$ and for $0 < p'' = a^s$ and $\alpha'' \in \left(\frac{\log p''}{\log a}, \frac{1-pN_a}{N_b}\right)$ with $\alpha'' = g(r, p'')$,

3. if $0 < p < a^s$ and $\alpha \in \left(\frac{\log p}{\log a}, \frac{\log p - p N_a}{\log b}\right)$, then $E^{(p)}_{\alpha} = \overline{F}(r_o) = E^{(p'')}_{\alpha''} = \overline{E}^{(p'')}_{\alpha''}$ for $0 < p' < a^s$ and $\alpha' = g(r, p')$ and for $a^s < p'' < \frac{1}{N_a}$ and $\alpha'' \in \left(\frac{\log p''}{\log a}, \frac{1-pN_a}{N_b}\right)$ with $\alpha'' = g(r, p'')$.

4. if $a^s < p < \frac{1}{N_a}$ and $\alpha \in \left(\frac{\log p}{\log a}, \frac{\log p - p N_a}{\log b}\right)$, then $E^{(p)}_{\alpha} = \overline{F}(r_o) = E^{(p'')}_{\alpha''} = \overline{E}^{(p'')}_{\alpha''}$ for $a^s < p < \frac{1}{N_a}$ and $\alpha' = g(r, p')$ and for $0 < p'' < a^s$ and $\alpha'' \in \left(\frac{\log p''}{\log a}, \frac{1-pN_a}{N_b}\right)$ with $\alpha'' = g(r, p'')$.

**Proof.** It is immediate from Theorem 3.3.  

**Remark 3.1** For $p = a^s$ where $N_a a^s + (N - N_a) b^s = 1$, $E^{(p)}_{\alpha} = F = \overline{E}^{(p)}_{\alpha}$ for $\alpha = s$ since

$$\frac{\log \mu_p(c_k(x))}{\log |c_k(x)|} = s$$

for each $k \in \mathbb{N}$ and we note that

$$\frac{\log p}{\log a} = \frac{\log 1-pN_a}{\log b} = s.$$ 

We note that for each $r \in (0, \frac{1}{N_a})$ we can relate $F(r)$ and $\overline{F}(r)$ with $E^{(p)}_{\alpha}$ and $\overline{E}^{(p)}_{\alpha}$ for some $p \neq a^s$ and some $\alpha$. In particular, for $0 < p < a^s$ and $a^s < p'' < \frac{1}{N_a}$, since $g(r, p)$ is continuous for $r \in [0, \frac{1}{N_a})$ and $g(0, p) = g(0, p'') < 0 < g(\frac{1}{N_a}, p) - g(\frac{1}{N_a}, p'')$, there exists $0 < r < \frac{1}{N_a}$ such that $g(r, p) = g(r, p'')$ by the intermediate value theorem. Let this value $g(r, p)$ be $\alpha$. Then $E^{(p)}_{\alpha} = F(r) = \overline{E}^{(p'')}_{\alpha''}$. We know that there exists unique such $r$ for $p$ and $p''$ since $g(r, p)$ is strictly increasing for $r \in (0, \frac{1}{N_a})$, and $g(r, p'')$
is strictly decreasing for \( r \in (0, \frac{1}{N_n}) \). It is just for the case (1) in the above corollary. Duality holds for
(2)-(4). From the above corollary, we also attain that \( E_{\alpha}^{(p)} = F(r) \) for \( p \neq a^s \), where \( E_{\alpha}^{(p)} = E_{\alpha}^{(p)} \cap E_{\alpha}^{(p)} \) and \( F(r) = F(r) \cap F(r) \), whereas \( F(a^s) \subset E_{s}(a^s) = F \) and \( F(a^s) \neq E_{s}(a^s) \).

**Corollary 3.5** Let \( s \) be the unique real number satisfying \( N_{a}s + (N - N_{a})b^s = 1 \) and let

\[
\delta(p) = \frac{pN_{a}\log p + (1 - pN_{a}) \log \frac{1 - pN_{a}}{N_{a}}}{pN_{a}\log a + (1 - pN_{a}) \log b}.
\]

Then
1. \( F(p) = E(p) \) if \( 0 < p < a^s \),
2. \( F(p) = E(p) \) if \( a^s < p < \frac{1}{N_{a}} \),
3. \( F(p) = E(p) \) if \( 0 < p < a^s \),
4. \( F(p) = E(p) \) if \( a^s < p < \frac{1}{N_{a}} \).

**Proof.** It is immediate from Theorem 3.3 with \( r = p \). ■

**Corollary 3.6** Let \( s \) be the unique real number satisfying \( N_{a}s + (N - N_{a})b^s = 1 \) and let

\[
\delta(p) = \frac{pN_{a}\log p + (1 - pN_{a}) \log \frac{1 - pN_{a}}{N_{a}}}{pN_{a}\log a + (1 - pN_{a}) \log b}.
\]

Then
1. \( \dim F(p) = \dim F(p) = \delta(p) \) and \( \dim F(p) = \delta(p) \) if \( 0 < p < a^s \),
2. \( \dim F(p) = \dim F(p) = \delta(p) \) if \( a^s < p < \frac{1}{N_{a}} \),
3. \( \dim F(a^s) = \dim F(a^s) = s \) and \( \dim F(a^s) = \dim F(a^s) = s \).

**Proof.** From the strong law of large numbers we obtain \( \mu_{p}(F(p)) = 1 \) where \( F(p) = F(p) \cap F(p) \). Applying [6,Proposition 2.3] to Corollary 3.5, we easily get \( \dim F(p) = \delta(p) \) and \( \dim F(p) = \delta(p) \) if \( 0 < p < a^s \). We also note that \( \delta(p) \leq \dim F(p) \leq \dim F(p) = \delta(p) \) if \( 0 < p < a^s \). Similarly (2) holds. For (3), \( F(a^s) \subset F(a^s) \) and \( F(a^s) \subset E_{s}(a^s) = F \), where \( E_{s}(a^s) = E_{s}(a^s) \cap E_{s}(a^s) \) and \( \mu_{a^s}(F(a^s)) = 1 \) by the strong law of large numbers. Similarly it holds for \( F(a^s) \). ■

**Corollary 3.7** Let \( p \in (0, \frac{1}{N_{a}}) \) and \( p \neq a^s \) where \( N_{a}s + (N - N_{a})b^s = 1 \). Let \( \alpha \) be in

\[
\left( \log \frac{p}{a}, \log \frac{1 - pN_{a}}{N_{a}} \right) \quad \text{or} \quad \left( \log \frac{1 - pN_{a}}{N_{a}}, \log \frac{p}{a} \right).
\]

For the solution \( r = r(\alpha) \) of the equation

\[
\alpha = \frac{N_{a}r \log p + (1 - N_{a}r) \log \frac{1 - pN_{a}}{N_{a}}}{N_{a}r \log a + (1 - N_{a}r) \log b}
\]

and \( \delta(r) = \frac{N_{a}r \log r + (1 - N_{a}r) \log \frac{1 - N_{a}r}{N_{a}}}{N_{a}r \log a + (1 - N_{a}r) \log b} \).

1. \( \dim E_{\alpha}^{(p)} = \dim E_{\alpha}^{(p)} = \dim E_{\alpha}^{(p)} = \dim F(r) = \delta(r) \),
2. \( \dim E_{\alpha}^{(p)} = \dim F(r) = \delta(r) \),
3. \( \dim E_{\alpha}^{(p)} = \delta(r) \) if \( 0 < p < a^s \) with \( a^s \leq r < \frac{1}{N_{a}} \) or \( a^s < p < \frac{1}{N_{a}} \) with \( 0 < r \leq a^s \),
4. \( \dim E_{\alpha}^{(p)} = \delta(r) \) if \( 0 < p < a^s \) with \( 0 < r \leq a^s \) or \( a^s < p < \frac{1}{N_{a}} \) with \( a^s \leq r < \frac{1}{N_{a}} \),
5. \( \dim E_{\alpha}^{(p)} = \dim E_{\alpha}^{(p)} = \dim E_{\alpha}^{(p)} = \dim E_{\alpha}^{(p)} = s \).

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Proof. For \( p(\neq a^s) \in (0, \frac{1}{N_a}) \) and \( \alpha \) in the assumption, we note that \( E_{\alpha}^{(p)} = E_{\alpha}^{(p)} \cap E_{\alpha}^{(p)} = F(r) \cap F(r) = F(r) \) where \( r = r(\alpha) \) of the equation

\[
\alpha = \frac{N_a r \log p + (1 - N_a r) \log \frac{1 - p N_a}{N_b}}{N_a r \log a + (1 - N_a r) \log b}.
\]

From Corollary 3.6, \( \delta(r) = \dim F(r) = \dim F(r) = \dim F(r) \) for every \( r \in (0, \frac{1}{N_a}) \). Since from Theorem 3.3, \( E_{\alpha}^{(p)} \) is \( F(r) \) or \( F(r) \) and \( E_{\alpha}^{(p)} \) is \( F(r) \) or \( F(r) \), where \( r = r(\alpha) \) of the equation

\[
\alpha = \frac{N_a r \log p + (1 - N_a r) \log \frac{1 - p N_a}{N_b}}{N_a r \log a + (1 - N_a r) \log b},
\]

so (1) follows. Similarly (2)-(5) follow from Theorem 3.3 and Corollary 3.6. ■

Remark 3.2 We find the Hausdorff and packing dimensions of \( F(r), F'(r), E_{\alpha}^{(p)}, E_{\alpha}^{(p)} \) and \( E_{\alpha}^{(p)} \) where \( r \in (0, \frac{1}{N_a}) \) and \( p \in (0, \frac{1}{N_a}) \) and

\[
\alpha \in \left( \frac{\log p}{\log a}, \frac{\log \frac{1 - p N_a}{N_b}}{\log b} \right) \text{ or } \left( \frac{\log \frac{1 - p N_a}{N_b}}{\log b}, \frac{\log p}{\log a} \right)
\]

except the information of upper bound of packing dimension about \( F(r) \) where \( 0 < r < a^s \) with \( N_a a^s + (N - N_a) b^s = 1 \), and \( F'(r) \) where \( a^s < r < \frac{1}{N_a} \), and \( E_{\alpha}^{(p)} \) where \( 0 < p < a^s \) with \( 0 < r(\alpha) < a^s \), or \( a^s < p < \frac{1}{N_a} \) with \( a^s < r(\alpha) < \frac{1}{N_a} \), and \( E_{\alpha}^{(p)} \) where \( 0 < p < a^s \) with \( a^s < r(\alpha) < \frac{1}{N_a} \), or \( a^s < p < \frac{1}{N_a} \) with \( 0 < r(\alpha) < a^s \).

Remark 3.3 We compare our results with those of [6, 8]. We note that, for every \( p \in (0, \frac{1}{N_a}) \) and \( p \neq a^s \) where \( N_a a^s + N_b b^s = 1, F(p) = E_{\alpha}^{(p)} \) where

\[
\delta(p) = \frac{p N_a \log p + (1 - p N_a) \log \frac{1 - p N_a}{N_b}}{p N_a (\log a) + (1 - p N_a) \log b},
\]

from which we see that \( \mu_p(E_{\alpha}^{(p)}) = 1 \) with the strong law of large numbers. Now Let \( \alpha \) be in

\[
\left( \frac{\log p}{\log a}, \frac{\log \frac{1 - p N_a}{N_b}}{\log b} \right) \text{ or } \left( \frac{\log \frac{1 - p N_a}{N_b}}{\log b}, \frac{\log p}{\log a} \right).
\]

Let \( \beta = q(\beta) \) as the positive number satisfying \( p N_a a^{\beta(q)} + N_b (\frac{1 - p N_a}{N_b}) b^{\beta(q)} = 1 \). We also note that

\[
E_{\alpha}^{(p)} = E_{q(\alpha) + \beta(q)},
\]

for every \( (q(\beta), q(\beta)) \). For \( (q(\beta), q(\beta)) \) satisfying the derivative \( \beta'(q) = -\alpha \) of \( \beta(\alpha) \) at \( q \).

\[
\mu_{q(\alpha) + \beta(q)}(E_{q(\alpha) + \beta(q)}) = 1 \text{ since } \delta(p q(\alpha) + \beta(q)) = q \alpha + \beta(q).
\]

Hence \( \dim E_{\alpha}^{(p)} = \dim E_{\alpha}^{(p)} = q \alpha + \beta(q) \) where \( p N_a a^{\beta(q)} + N_b (\frac{1 - p N_a}{N_b}) b^{\beta(q)} = 1 \) and \( \beta'(q) = -\alpha \).

Remark 3.4 A self-similar set \( F \) is completely decomposed into classes by the lower and upper dimension sets as \( F = \bigcup_{0 < r < \frac{1}{N_a}} F_{\alpha}^{(p)} \) and \( F = \bigcup_{0 < r < \frac{1}{N_a}} F_{\alpha}^{(p)} \). Then \( F \) is partitioned into \( F_1(r) \) by the equivalence relation that \( x \) is equivalent to \( y \) if \( x, y \in F(r) \) for some \( r \in (0, \frac{1}{N_a}) \). Similarly, \( F \) is partitioned into \( F_2(r) \) by the equivalence relation that \( x \) is equivalent to \( y \) if \( x, y \in F(r) \) for some \( r \in (0, \frac{1}{N_a}) \). Fix \( p(\neq a^s) \in (0, \frac{1}{N_a}) \) where \( N_a a^s + (N - N_a) b^s = 1 \). Since \( g(r, p) \) is strictly monotone for \( r \in (0, \frac{1}{N_a}) \) from

Remark 3.1, the function \( g(r, p) \) of \( r \) is a one-to-one correspondence between \( (0, \frac{1}{N_a}) \) and \( \left( \frac{\log p}{\log a}, \frac{\log \frac{1 - p N_a}{N_b}}{\log b} \right) \)

or \( \left( \frac{\log \frac{1 - p N_a}{N_b}}{\log b}, \frac{\log p}{\log a} \right) \). Then \( F \) is completely decomposed into the spectral classes by the lower and upper local dimensions of a self-similar measure \( \mu_p \) as

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\[ F = \bigcup_{\alpha \in \left( \frac{\log 1}{\log a}, \frac{\log pNa}{\log b} \right)} E^{(p)}_{\alpha} = \bigcup_{0<r<\frac{1}{Na}} F_1(r) \quad \text{if} \quad 0 < p < a^s, \]

\[ F = \bigcup_{\alpha \in \left( \frac{\log pNa}{\log b}, \frac{\log 1}{\log a} \right)} E^{(p)}_{\alpha} = \bigcup_{\frac{1}{Na}<r<1} F_2(r) \quad \text{if} \quad a^s < p < \frac{1}{Na}, \]

\[ F = \bigcup_{\alpha \in \left( \frac{\log 1}{\log a}, \frac{\log pNa}{\log b} \right)} E^{(p)}_{\alpha} = \bigcup_{0<r<\frac{1}{Na}} F_2(r) \quad \text{if} \quad 0 < p < a^s, \]

\[ F = \bigcup_{\alpha \in \left( \frac{\log pNa}{\log b}, \frac{\log 1}{\log a} \right)} E^{(p)}_{\alpha} = \bigcup_{\frac{1}{Na}<r<1} F_1(r) \quad \text{if} \quad a^s < p < \frac{1}{Na}. \]

We note that all spectral classes by the lower or upper local dimensions of all self-similar measures, except for the singular self-similar measure having the exact dimension of the Hausdorff and packing dimension of the self-similar set, are just classified into two classes: the lower distribution class and the upper distribution class.

4 Conclusion

We relate the distribution sets in the distribution sets with the subsets in a spectral class of a self-similar set. Then we compute their respective Hausdorff and packing dimensions and compare our results with those of others. Finally, we attain that all the spectral classes by the self-similar measures and their lower and upper local dimensions are classified into two classes: the lower distribution class and the upper distribution class.

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References


