Nonwandering Operator Sequences in Banach Space

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Abstract: In this paper, we study the nonwandering operator sequences in infinite dimensional separable Banach space. Based on the definition of nonwandering operator in Banach space and the definition of the nonwandering operator semigroups of the PDE in special Banach space, we give the definition of nonwandering operator sequences and show the existence of it in infinite dimensional separable Banach space of analytic function of several complex variables. Finally, we present some properties of nonwandering operator sequences.

Keywords: Nonwandering operator sequence; Runge domain; Banach space of analytic function; Exponential type entire function

1 Introduction

The linear operators in finite dimensional space never have chaotic property and only nonlinear operators on finite dimensional space may cause chaos. However only the linear operators in infinite dimensional space could possess the strange chaotic property, which are called linear chaotic operators ([1]). Since 1990, it has attracted wide attention. Tian introduced a new type of linear chaotic operators—nonwandering operator in some typical infinite Banach spaces, including Banach sequence space and physical background space and he also gave the nonwandering operator semigroups of PDE in special Banach space ([16]).

Nonwandering operator is not only relative to the problem of invariant subspace, hypercyclic operator, supercyclic operator and cyclic operator in functional analysis, but also relative to the linear chaotic operator in infinite dimensional space, Axiom A system. In the research of non-trivial invariant subspace, we note that bounded linear operators never have non-trivial invariant subspace (or subset) if and only if all the nonzero vectors are cyclic (or hypercyclic). It naturally connects the research of invariant subspace (or subset) to that of cyclicity or hypercyclicity of bounded linear operators. C. A. Read, P. Enflo, and A. Atzmon ([7]-[10]) found that in Banach space, there exactly exist some operators that have no non-trivial closed invariant subspaces. However, for the bounded linear operators in infinite dimensional Hilbert space, we don’t know whether there exists non-trivial closed invariant subspace (or subset), which is an open problem and needs some new solutions. The hypercyclic decomposition of nonwandering operators is given in [15]. The work of nonwandering operator and nonwandering operator sequence are relative to hypercyclic operator, or non-trivial invariant subspace and subset. We can get some examples in which one operator is nonwandering operator and also hypercyclic operator. But it does not mean that it is the same as hypercyclic operator. When nonwandering operator $T$ is invertible, we have the following properties: $\sigma(T) \cap \partial D = \phi$. Yet, $T$ is a hypercyclic operator iff $\sigma(T) \cap \partial D \neq \phi$. Hence the two kinds of operators are different from each other when they are invertible. However we can give out some concrete examples characterizing their relationship.

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The first observation of a hypercyclic operator was done by G. D. Birkhoff in 1929([11]). Let \( H(C) \) be the space of entire functions on the complex plane, endowed with the compact-open topology. Birkhoff showed the hypercyclicity of the translation operators \( T_\alpha f(z) = f(z + \alpha), \alpha \neq 0, f(z) \in H(C), z \in C \). The translation operators defined by Birkhoff can also be viewed as differentiation operators, that is \( T_\alpha = e^{\alpha D} \), where \( D \) is a differentiation operator which satisfies \( T_\alpha f(z) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} D^n f \) for each \( f \in H(C) \), (convergence in \( H(C) \)). From this perspective, G. R. MacLane([12]) showed in 1952 that differentiation operator \( D : f \rightarrow Df \) was also hypercyclic. In 1991, for the first time, G. Godefroy and J. Shapiro [1] connected the research of hypercyclic operator with linear chaotic operator and obtained that some hypercyclic operators are chaotic under the definition of Devaney ([13]). Furthermore they generalized Birkhoff’s and Maclane’s result and showed that every continuous linear operator (not a multiple of the identity) on \( H(C) \) that commutes with differentiation operator is hypercyclic. Luis Bernal-Gonzalez proved that subexponential type entire function sequences of Differential operators is hypercyclic sequences on \( H(\mathbb{C}) \). From this perspective, G. R. MacLane([12]) showed in 1952 that differentiation operator \( D : f \rightarrow Df \) was also hypercyclic. In 1991, for the first time, G. Godefroy and J. Shapiro [1] connected the research of hypercyclic operator with linear chaotic operator and obtained that some hypercyclic operators are chaotic under the definition of Devaney ([13]). Furthermore they generalized Birkhoff’s and Maclane’s result and showed that every continuous linear operator (not a multiple of the identity) on \( H(C) \) that commutes with differentiation operator is hypercyclic. Luis Bernal-Gonzalez proved that subexponential type entire function sequences of Differential operators is hypercyclic sequences on \( H(\mathbb{C}) \).

Moreover, we give some properties of nonwandering operator sequences at the end of this paper.

## 2 Basic notations and definitions

**Definition 2.1** ([6]) Suppose \( T \in L(X) \), if

(1) there exists a closed subspace \( E \subset X \), which has hyperbolic structure:

\[
E = E^u \oplus E^s, \quad T E^u = E^u, \quad T E^s = E^s,
\]

where \( E^u, E^s \) are closed subspaces. In addition, there exists constants \( \tau (0 < \tau < 1) \) and \( c > 0 \), such that

\[
\| T^k \xi \| \geq c \tau^{-k} \| \xi \|, \quad \forall \xi \in E^u, k \in \mathbb{N};
\]

\[
\| T^k \eta \| \leq c \tau^k \| \eta \|, \quad \forall \eta \in E^s, k \in \mathbb{N};
\]

(2) \( \text{Per}(T) \) is dense in \( E \), then \( T \) is said to be a nonwandering operator relative to \( E \) in \( X \).

**Definition 2.2** ([16]) Suppose \( T(t) \in L(X) \) is an operator semigroup on \( X \), if

(1) there exists a closed subspace \( E \subset X \), which has hyperbolic structure:

\[
E = E^u \oplus E^s, \quad T(t) E^u = E^u, \quad T(t) E^s = E^s
\]

where \( E^u, E^s \) are closed subspaces. In addition, there exist constants \( \tau (0 < \tau < 1) \) and \( c > 0 \), such that for each \( \xi \in E^u, \eta \in E^s, t \geq 0 \), the inequalities

\[
\| T(t) \xi \| \geq c \tau^{-t} \| \xi \|, \quad \| T(t) \eta \| \leq c \tau^t \| \eta \| \quad \text{holds}.
\]

(2) \( \text{Per}(T(t)) \) is dense in \( E \), then \( \{ T(t) \}_{t \geq 0} \) is called a nonwandering operator semigroup relative to \( E \) in \( X \).

Based on the definition 2.1 and 2.2, we give the definition of nonwandering operator sequence as follows:

**Definition 2.3** Suppose \( \{ T_n : n \in \mathbb{N} \} \) is an linear operator sequence on \( X \), if

(1) there exists a closed subspace \( E \subset X \), which has hyperbolic structure:

\[
E = E^u \oplus E^s, \quad T_n E^u = E^u, \quad T_n E^s = E^s
\]

where \( E^u, E^s \) are closed subspaces. In addition, there exist constants \( \tau (0 < \tau < 1) \) and \( c > 0 \), such that for each \( \xi \in E^u, \eta \in E^s, t \geq 0 \) holds

\[
\| T_n \xi \| \geq c \tau^{-n} \| \xi \|, \quad \| T_n \eta \| \leq c \tau^n \| \eta \| \ .
\]

(2) \( \text{Per}(T_n) \) is dense in \( E \);
then we call \( \{ T_n : n \in \mathbb{N} \} \) a nonwandering operator sequence relative to \( E \) in \( X \).

**Definition 2.4** ([2]) Let \( G \) be a nonempty open subset of \( C^N \). \( G \) is said to be a domain when it is connected. A domain \( G \subset C^N \) is said to be a Runge domain if each analytic function on \( G \) can be approximated uniformly by polynomials on every compact subset of \( G \). When \( N = 1 \), the Runge domains are precisely the simply connected domains. Denoted by \( H (G) \), as usual, the analytic functions on \( G \) endowed with the compact-open topology.

**Definition 2.5** ([1]) For \( 1 \leq j \leq N \), let \( D_j \) denote complex partial differentiation with respect to the \( j \)th coordinate. A multi-index is an \( N \)-tuple \( p = (p_1, \ldots , p_N) \) of nonnegative integers. Denote \( |p| = p_1 + \cdots + p_N, |p| = p_1 ! \cdots p_N ! \), \( D^p = D_1^{p_1} \circ \cdots D_N^{p_N} (D^0 = I = \text{the identity operator}) \), and \( z^p = z_1^{p_1} \cdots z_N^{p_N} \) if \( z = (z_1, \ldots , z_N) \). An entire function \( \Phi(z) = \sum_{|p|\geq 0} a_p z^p \) on \( C^N \) is said to be of exponential type whenever there exist positive constants \( \Phi, \lambda, a, b \) and \( N \) such that \( \Phi(z) \leq Ae^{B|z|} (z \in C^N) \). This happens if and only if there is \( R \in (0, +\infty) \) for which \( |a_p| \leq \frac{R^p}{p!} (|p| \geq 0) \).

We say that an entire function \( \Phi(z) = \sum_{|p|\geq 0} a_p z^p \) on \( C^N \) is of subexponential type if for \( \varepsilon > 0 \), there exists a positive constant \( K = K(\varepsilon) \) such that \( |\Phi(z)| \leq Ke^{\varepsilon|z|} (z \in C^N) \). A straightforward computation with power series and the Cauchy inequalities ([14]) shows that \( \Phi \) is of subexponential type if and only if, given \( \varepsilon > 0 \), there is a positive constant \( A = A(\varepsilon) \) such that

\[
|a_p| \leq A \frac{e^{\varepsilon|p|}}{p!} (|p| \geq 0).
\]

Note that, if \( N = 1 \), then \( \Phi \) is of subexponential type if and only if \( \Phi \) is either of growth order less than one or of growth order one and growth type zero. Each entire function of subexponential type is obviously of exponential.

**Definition 2.6** ([2]) A sequence \( \{ \Phi_n \}_{n=1}^\infty \) of entire functions on \( C^N \) is said to satisfy condition \((P)\) if:

There are two nonempty open subsets \( A, B \) of \( C^N \) such that for every pair of finite subset \( F_1 \subset A \) and \( F_2 \subset B \), there exists a subsequence \( \{ n_k \} \) of positive integers such that \( \lim_{k \to \infty} \Phi_{n_k}(a) = 0 \) for each \( a \in F_1 \), and \( \lim_{k \to \infty} \Phi_{n_k}(b) = 0 \) for each \( b \in F_2 \).

### 3 Nonwandering operator sequences in Banach space

#### 3.1 An example of nonwandering operator sequences in Banach space of analytic function

Let \( X \) be an infinite dimensional Banach space of analytic function. We consider the sequence \( \Phi_n(z) = z^n (z \in C^N, n \in \mathbb{N}) \) on \( X \), and it is obvious that \( \Phi_n \) is the function of subexponential type on \( X \). Moreover \( \{ \Phi_n, n \in \mathbb{N} \} \) satisfies condition \((P)\) (only set \( A = \{ z : |z| < 1 \}, B = \{ z : |z| > 1 \} \)). Next we prove that sequences of differential operators \( \{ \Phi_n(D), n \in \mathbb{N} \} \) (\( D \) is differential operators) is nonwandering operator sequences on \( X \).

Firstly, we construct a closed invariant subspace \( E \subset X \) such that \( E \) has hyperbolic structure.

For \( \Phi_n (a) \in R, a \in A \), suppose \( \{ e_1 \}_{i=0}^\infty \) is the basis of \( X \), and let \( y_0 = \sum_{i=0}^\infty e_i e_{a_i} (a_i \in A) \) such that \( \Phi_n(D)y_0 = \Phi_n (a_i) y_0 \). Then we get a vector \( y_1 = \{ e_{a_1}, a_1 e_{a_1}, a_1^2 e_{a_1}, \ldots \ldots \} \), \( a_1 \in A \).

Let \( E^s = \text{span} \{ y_1 \} \), we can easily obtain that \( E^s \) is eigenvector closed invariant subspace for every \( \Phi_n (D) = D^n \) with eigenvalue \( \Phi_n (a_1) \). Furthermore, for each \( \chi \in E^s \), we have \( \chi = \lambda y_0 \), such that

\[
||\Phi_n(D)\chi|| = ||\lambda \Phi_n (a_1) y_0|| = ||\Phi_n (a_1)|| \||\chi\||, \quad ||\Phi_n(a_1)|| = |a_1^n| < 1
\]

Similarly, we set \( y_1 = \sum_{i=0}^\infty e_i e_{b_i}, \) and we get a vector:

\[
y_2 = \{ e_{b_1}, b_1 e_{b_1}, b_1^2 e_{b_1}, \ldots \ldots \} , b_1 \in B
\]
Let $E^n = \text{span} \{y_2\}$, and we can easily obtain that $E^n$ is eigenvector closed invariant subspace for every $\Phi_n(D) = D^n$ with eigenvalue $\Phi_n(b_1)$. Furthermore, for each $y \in E^n$, we have $y = \lambda y_1$, such that
\[
\|\Phi_n(D) y\| = \|\lambda \Phi_n(b_1) y_1\| = |\Phi_n(b_1)| \|y\|, \quad |\Phi_n(b_1)| = |b_1^n| > 1
\]
Set $E = E^s \oplus E^u, \tau(\max(|\Phi_n(a_1)|, |\Phi_n(b_1)|^{-1}) < \tau < 1)$, and we get
\[
\|\Phi_n(D) \xi\| \geq \tau^{-n} \|\xi\|, \quad \forall \xi \in E^n, n \in N
\]
\[
\|\Phi_n(D) \eta\| \leq \tau^n \|\eta\|, \quad \forall \eta \in E^s, n \in N
\]
Secondly, we prove that $\text{Per}(\Phi_n(D)) = \text{Per}(D^n)$ is dense in $E$.

Suppose $e_a$ is eigenvalue for $\Phi_n(a)$, that is $\Phi_n(D) e_a = \Phi_n(a) e_a$, where $\Phi_n(a)$ is root of unit. And then, by the theorem 3.5 of [16], $\text{Per}(\Phi_n(D))$ is dense in $E$. Hence, $\{\Phi_n(D), n \in N\}$ is nonwandering operator sequence on $X$.

3.2 Nonwandering sequences in Banach space

Motivated by the above result, we consider the question whether the sequences of Differential operators on infinite dimensional separable Banach space of analytic function, which satisfy condition (P), are nonwandering operator sequences. The answer is positive. We give a lemma first for the sake of completeness.

**Lemma 3.1** ([12]) If $G \subset C^N$ is a nonempty open subset and $\Phi(Z) = \sum_{|p| \geq 0} a_p z^p$ is an entire function of subexponential. Then the series $\Phi(D) = \sum_{|p| \geq 0} a_p z^p$ defines an operator on $H(G)$.

**Proof.** If $G = C^N$, the result is a particular case. So we may consider that $G \neq C^N$. Fix a function $f \in H(G)$, and a compact subset $K \subset G$. Let $\varepsilon = \frac{1}{2} d(K, C^N / G)$. Then there is $A \in (0, \infty)$ such that $|a_p| \leq A.((\varepsilon / 2)^{|p|}/|p|)(|p| \geq 0)$. Fix a point $a \in K$, the Cauchy formula for derivatives [15] tells that
\[
|D^p f(a)| \leq \frac{|p! \|f\|_{D(a, \varepsilon)}}{\varepsilon^{|p|}} \leq \frac{|p! \|f\|_{K_1}}{\varepsilon^{|p|}},
\]
where $K_1$ is the compact set $\{z : d(z, K) \leq \varepsilon\}$. Note that $K \subset K_1 \subset G$, therefore
\[
\|D^p f\|_{K} \leq p! \|f\|_{K_1} / \varepsilon^{|p|},
\]
\[
\sum_{|p| \geq 0} a_p D^p f \|_{K} \leq \sum_{|p| \geq 0} A.((\varepsilon / 2)^{|p|}/|p|).(|p| \|f\|_{K_1} / \varepsilon^{|p|}) = 2^N A. \|F\|_{K_1},
\]
so the series $\sum_{|p| \geq 0} a_p D^p f$ converges uniformly on $K$ and $\Phi(D)$ defines a mapping from $H(G)$ into itself. The linearity of it is trivial and, since $\|\Phi(D) f\|_{K} \leq 2^N A. \|f\|_{K_1}, \forall f \in H(G)$, we obtain that $\Phi(D)$ is continuous on $H(G)$.

**Theorem 3.1** Suppose that $G$ is a Runge domain of $C^N$ and $\Phi_n(n \in N)$ are entire functions on $C^N$. Assume that $\Phi$ is not a constant and denote $L_n = \Phi_n(D)(n \in N)$. Suppose that every $\Phi_n$ is of subexponential (exponential, resp) type and the sequence $\{\Phi_n, n \in N\}$ satisfy condition (P), then operator sequence $\{L_n, n \in N\} = \{\Phi_n(D), n \in N\}$ are nonwandering operator sequences.

**Proof.** By lemma 3.1, every $L_n$ is a operator defined on $H(G)$ (even on $H(C^N)$) if $\Phi_n$ is of exponential type). From now on, $G$ may be $C^N$ (when $\Phi_n$ is of exponential type) or not (when $\Phi_n$ is of is of exponential type). Note that for each $j \in \{1, 2, 3, \ldots, N\}$, we have $D_j e_a = a_j e_a (\forall a \in C^N)$. So for every multi-index $P, D^p e_a = a^p e_a$. Then for each $a \in C^N, n \in N$, $L_n e_a = \Phi_n(D) e_a = \Phi_n(a) e_a$. Observe that each function $e_a$ is an eigenvector for every $L_n$ with eigenvalue $\Phi_n(a)$.

Firstly, we construct a closed invariant subspace $E \subset X$ such that $E$ has hyperbolic structure.

Since $\Phi_n$ satisfies condition(P), there is subset $A$ of $C^N$ such that for every finite subset $F_1$ there is sequence of positive integers $\{n_k\}$, such that
\[
\lim_{k \to \infty} \Phi_{n_k}(a) = 0 (\forall a \in F_1)
\]

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Set finite subset $F_{11} \subset A$, and then there is subsequence of positive integers $\{n_{k1}\}$, such that for each $a \in F_{11},$

$$\lim_{k1 \to \infty} \Phi_{n_{k1}}(a) = 0.$$ 

Set finite subset $F_{12} \subset A, (F_{11} \cap F_{12} \neq \Phi)$, and also there is subsequence of positive integers $\{n_{k2}\}$, such that $\forall a \in F_{12},$

$$\lim_{k2 \to \infty} \Phi_{n_{k2}}(a) = 0.$$ 

Moreover if we take $\omega_1 = F_{11} \cap F_{12}$, and then for each $a \in \omega_1,$

$$\lim_{k1 \to \infty} \Phi_{n_{k1}}(a) = 0, i = 1, 2.$$ 

If we let finite subset $F_{13} \subset A (\omega_1 \cap F_{13} \neq \Phi)$, there is subsequence of positive integers $\{n_{k3}\}$, such that for each $a \in F_{13},$

$$\lim_{k3 \to \infty} \Phi_{n_{k3}}(a) = 0.$$ 

Consider $\omega_2 = \omega_1 \cap F_{13}$, then for each $a \in \omega_2,$

$$\lim_{k1 \to \infty} \Phi_{n_{k1}}(a) = 0, i = 1, 2, 3.$$ 

Go on with the course till we find finite subset $F_{1p} \subset A$, and there is subsequence of positive integers $\{n_{kp}\}$, such that for each $a \in F_{1p},$

$$\lim_{kp \to \infty} \Phi_{n_{kp}}(a) = 0$$

Take $\omega_{p-1} = \omega_{p-2} \cap F_{1p}$, and then for each $a \in \omega_{p-1},$

$$\lim_{k1 \to \infty} \Phi_{n_{k1}}(a) = 0 \quad i = 1, 2, 3, \ldots , p$$

Such that

$$\{n_{ki}, i = 1, 2, \ldots, p \} = \{n, n \in N \}$$

Consider $E_1 = \bigcap_{i=1}^{p} F_{1i} = \omega_{p-1}$, and then for each $a \in E_1,$

$$\lim_{n \to \infty} \Phi_n(a) = 0, n \in N.$$ 

For each $\Phi_n(a) \in R(\Phi(a) \in E_1)$, by the initial considerations, $e_a (a \in E_1)$ is eigenvector with eigenvalue $\Phi_n(a)$, that is

$$L_n e_a = \Phi_n(D)e_a = \Phi_n(a)e_a.$$ 

Let $E'_1 = \bigg\{ \eta | \eta = \sum_{i=0}^{\infty} \alpha_i e_{a_i}, \Phi_n(D)e_{a_i} = \Phi_n(a_i)e_{a_i}, \sum_{i=0}^{\infty} \alpha_i e_{a_i} \in E, \forall n \in N, \Phi(a_i) \leq \tau_p \bigg\}$

and $E^s = \overline{E_1}$, Hence $E^s \subset H(G)$ is a closed subspace, and for each $\eta = \sum_{i=0}^{\infty} \alpha_i e_{a_i} \in E^s, n \in N$. we have

$$\|L_n \eta\| = \|\Phi_n(D) \eta\| = \|\Phi_n(D) \sum_{i=0}^{\infty} \alpha_i e_{a_i}\| = \|\sum_{i=0}^{\infty} \alpha_i \Phi_n(a_i) e_{a_i}\|$$

$$= \max_{z \in G} \left| \sum_{i=0}^{\infty} \alpha_i \Phi_n(a_i) e_{a_i} z \right| \leq \tau^p \max_{z \in G} \left| \sum_{i=0}^{\infty} \alpha_i e_{a_i} z \right| = \tau^p \|\eta\|$$

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It is similar to consider the nonempty open subset \( B \) of \( C^N \). By the above method we can find \( E_2 = \bigcap_{i=1}^m E_{2i} \), for each \( b \in E_2 \), \( n \in N \), and \( \lim_{n \to \infty} \Phi_n (b) = \infty \). Let

\[
E_2' = \left\{ \xi \mid \xi = \sum_{i=0}^{\infty} \beta_i e_{b_i}, \Phi_n (D) e_{b_i} = \Phi_n (b_i) e_{b_i}, \exists 0 < \tau_2 < 1, \forall n \in N, \Phi_n (b_i) \geq \tau_1^{-n} \right\}
\]

and \( E'' = \overline{E_2} \).

Hence \( E'' \subset H (G) \) is a closed subspace, and for each \( \xi = \sum_{i=0}^{\infty} \beta_i e_{b_i}, n \in N \), we have

\[
\| L_n \xi \| = \| \Phi_n (D) \xi \| = \| \Phi_n (D) \sum_{i=0}^{\infty} \beta_i e_{b_i} \| = \| \sum_{i=0}^{\infty} \beta_i (\Phi_n (b_i) e_{b_i}) \| = \max_{z \in G} \left| \sum_{i=0}^{\infty} \beta_i (\Phi_n (b_i) e_{b_i}) \right| \geq \tau_2^{-n} \max_{z \in G} \left| \sum_{i=0}^{\infty} \beta_i e_{b_i} \right| = \tau_2^{-n} \| \xi \| .
\]

If we set \( \tau = \max (\tau_1, \tau_2) \), then for

\[
\forall \eta \in E^s, n \in N, \| L_N \eta \| \leq \tau^n \| \eta \| ,
\]

\[
\forall \xi \in E'', n \in N, \| L_n \xi \| \geq \tau^{-n} \| \xi \| .
\]

In the following, we’ll prove that \( E'' \) and \( E^s \) are invariants under the operator \( L_n \).

For each \( \xi \in E'' \),

\[
\xi = \sum_{i=0}^{\infty} \beta_i e_{b_i} = \Phi_n (D) \sum_{i=0}^{\infty} \left( \frac{\beta_i}{\Phi_n (b_i)} \right) e_{b_i} \subset \Phi_n (D) E'' = L_n E'' ,
\]

hence \( E'' \subset L_n E'' \). On the other hand, for each \( \Psi \in L_n E'' \), there is \( \xi = \sum_{i=0}^{\infty} \beta_i e_{b_i} \in E'' \), such that \( L_n \xi = \Psi \).

Hence \( \Psi = L_n \left( \sum_{i=0}^{\infty} \beta_i e_{b_i} \right) = \Phi_n (D) \left( \sum_{i=0}^{\infty} \beta_i e_{b_i} \right) = \left( \beta_n \Phi_n (b_i) e_{b_i} \right) \in E'' \). So \( L_n E'' \subset E'' \) and \( L_n E'' = E'' \). It is similar to prove \( L_n E^s = E^s \). Set \( E = E'' \oplus E^s \), and we obtain that \( E \) is a closed invariant subspace that has hyperbolic structure on \( H (G) \).

Secondly, we need to prove that \( Per (L_n) \) is dense in \( H (G) \).

Since \( G \) is Runge domain, we set a compact subset \( K \subset G \). By the theorem 5.1, property 5.2 and theorem 6.2 of [1], \( Per (L = \Phi_n (D)) \) is dense in \( H (G) \). That is to say for \( \forall f \in H (G) \), \( \forall \varepsilon > 0 \), there exists \( g \in Per (\Phi (D)) \), such that for \( \forall Z \in K \),

\[
| f (z) - g (z) | < \frac{\varepsilon}{2}
\]

On the other hand, since \( \{ \Phi_n (D) \} \) converges uniformly on every compact subset \( K \) of \( G (11) \), that is the sequences of operators \( \{ \Phi_n (D) \} \) converges pointwise on \( H (G) \), and \( \lim_{n \to \infty} \Phi_n (D) = \Phi (D) \). Therefore for \( \forall g \in Per (\Phi (D)) \), there exists \( g_n \in Per (\Phi_n (D)) \), such that for \( \forall \varepsilon > 0, \forall z \in K \), we have

\[
| g (z) - g_n (z) | < \frac{\varepsilon}{2}
\]

So for \( \forall \varepsilon > 0, z \in K \), we have

\[
| f (z) - g_n (z) | < \varepsilon
\]

Then the result that \( Per (\Phi_n (D)) \) is dense in \( H (G) \) is obtained.

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4 Properties of nonwandering operator sequence

In the sense of the properties of nonwandering operators, we easily show:

Proposition 4.1 Suppose \( L_n \in L(X, \| \cdot \|) \) and \( E \subset X \) is a closed subspace, then \( L_n \) is a nonwandering operator sequence relative to \( E \) if and only if the following conditions hold:

1. \( E = E_u \oplus E_s, L_n E_u = E_u, L_n E_s = E_s \), and there exists some norm \( \| \cdot \| \), which is equivalent to \( \| \cdot \| \), such that
   \[
   \| L_n^u \| = \| L_n \|_{E_u} > 1, \| L_n^s \| = \| L_n \|_{E_s} < 1.
   \]

2. \( \text{Per}(L_n) \) is dense in \( E \).

Proposition 4.2 Let \((X, \| \cdot \|)\) be an infinite dimensional separable Banach space, and \( E_1, E_2 \) be closed subspaces in \( X \) and \( E_1 \cap E_2 = \{0\} \). If the restrictions \( L_n |_{E_1}, L_n |_{E_2} \in L(X) \) are invertible nonwandering operator sequence relative to \( E_1, E_2 \) respectively, then \( L_n |_{E} \) is a nonwandering operator sequence relative to \( E = E_u \oplus E_s \).

Proof. Since \( L_n |_{E_i}, (i = 1, 2) \) is a nonwandering operator sequences relative to \( E_i \), then \( E_i \) has hyperbolic structure:

\[
E_i = E_i^u \oplus E_i^s, TE_i^u = E_i^u, TE_i^s = E_i^s,
\]

\( E_u^i, E_s^i \) are also closed subspaces. Furthermore, there exist \( 0 < \tau_i < 1 \) and constant \( \tau_i > 0 \), such that for each \( \xi_i \in E_i^u, \eta_i \in E_i^s, k \in N \),

\[
\| L_n^{−k} \xi_i \| \leq c_1 \tau_i^k \| \xi_i \|, \quad \| L_n^k \eta_i \| \leq c_1 \tau_i^k \| \eta_i \|,
\]

and \( \text{Per}(L_n) \) is dense in \( E_i \).

Let \( E_u = E_1^u \oplus E_2^u \), and we define the following norm on \( E_u \): \( \forall x \in E_u, x = x_1 + x_2, x_1 \in E_1^u, x_2 \in E_2^u, \| x \|_0 = \max\{ \| x_1 \|, \| x_2 \| \} \), then \( \| \cdot \|_0 \) is equivalent to \( \| \cdot \| \) (see Lemma 5.3). Namely, there exist constants \( c_1 > 0 (i = 3, 4) \), such that \( c_4 \| x \| \leq \| x \|_0 \leq c_3 \| x \|, \forall x \in E_u \).

For each \( x \in E_u, x = x_1 + x_2, x_1 \in E_1^s, x_2 \in E_2^s \),

\[
\| L_n^{−k} x \| = \| L_n^{−k} (x_1 + x_2) \| \leq \| L_n^{−k} x_1 \| + \| L_n^{−k} x_2 \| \leq c_1 \tau_1^k \| x_1 \| + c_2 \tau_2^k \| x_2 \| \leq c_1 \tau_1^k \| x \|_0 + c_2 \tau_2^k \| x \|_0 \leq c_1 c_3 \tau_1^k \| x \| + c_2 c_3 \tau_2^k \| x \| .
\]

Let \( c = \max\{c_1, c_2, c_3, c_4\}, \tau = \max\{\tau_1, \tau_2\} \), then \( c > 0, 0 < \tau < 1 \), and \( \| L_n^{−k} x \| \leq 2 c \tau^k \| x \| \) for each \( x \in E_u, k \in N \). Let \( E_s = E_1^s \oplus E_2^s \), then for each \( y \in E_s, k \in N \), \( \| L_n^k y \| \leq 2 c \tau^k \| y \| \). Thereby,

\[
E = E_u \oplus E_s, L_n E_u = L_n E_1^u \oplus L_n E_2^u = E_u^s, L_n E_s = L_n E_1^s \oplus L_n E_2^s = E_u^s \oplus E_s^u = E_s^u \oplus E_s^u = E_s^s.
\]

Since \( \text{Per}(L_n) \) is dense in \( E_1, E_2 \) respectively, \( \text{Per}(L_n) \) is dense in \( E_1 \oplus E_2 = E_u \oplus E_s = E \). Thus \( L_n |_{E} \) is nonwandering operator sequence relative to \( E \).

References


*IJNS homepage: http://www.nonlinearscience.org.uk/*