Some Exact Complex Solutions of the Klein-Gordon-Shrödinger Equations

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Abstract: In this paper, some complex solutions are obtained for the Klein-Gordon-Schrödinger equations by the \((G'/G)\)-expansion method with arbitrary parameters. The obtained solutions are hyperbolic, trigonometric and rational functions. When the parameters are taken as special values, it is observed that the previously known solutions can be recovered. The procedure actually derives many complex solutions in a simple way.

Keywords: \((G'/G)\)-expansion method; Klein-Gordon-Schrödinger equations; Traveling wave complex solutions.

1 Introduction

In the nonlinear sciences, it is well known that many nonlinear partial differential equations (PDEs) are widely used to describe the complex phenomena. The powerful and efficient methods for finding analytic solutions of nonlinear equations have got a lot of attention by a diverse group of scientists. Some examples are, the homogeneous balance method [1], the Exp-function method [2, 3], the Sine-Cosine function method [4], the Tanh-Coth function method [5], the first integral method [6].

Recently, Wang et al. [7] introduced an expansion technique called the \((G'/G)\)-expansion method and demonstrated that it is a powerful technique for seeking analytic solutions for nonlinear partial differential equations. Furthermore, Bekir [8] applied this method to obtain traveling wave solutions of some nonlinear evolution equations. More recently, some useful studies by the authors [9–11] also appeared in the literature.

In the present paper, we will extend the \((G'/G)\)-expansion method to the coupled Klein-Gordon-Schrödinger (K-G-S) equations:

\[
\begin{align*}
\psi_t + \frac{1}{2} \Delta \psi + \phi \psi &= 0 \\
\phi_{tt} - \Delta \phi + M^2 \phi - |\psi|^2 &= 0
\end{align*}
\]

which describes a system of conserved scalar nucleons interacting with neutral scalar mesons. Here, \(\psi\) represents a complex scalar nucleon field and \(\phi\) a real scalar meson field. The real constant \(M\) describes the mass of a meson and \(\Delta = \frac{\partial^2}{\partial x^2}\). See [12] for more details on this system.

Much work has been done on the existence of global solutions, asymptotic behavior and stability of the discussed problem [12–14]. In particular, many diverse methods have been implemented for searching explicit exact solutions, such as the homogeneous balance method [1] and the Exp-function method [3].

2 Description of the \((G'/G)\)-expansion method

We suppose that the given nonlinear partial differential equation for \(u(x, t)\) is in the form

\[
P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0,
\]

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where $P$ is a polynomial. The essence of the $\left(\frac{G'}{G}\right)$-expansion method can be presented in the following steps:

**step 1.** Seek traveling wave solutions of Eq. (3) by taking $u(x,t) = U(\zeta)$, $\zeta = x - ct$, and transform Eq. (3) to the ordinary differential equation

$$Q(U, U', U'', \ldots) = 0,$$

(4)

where prime denotes the derivative with respect to $\zeta$.

**step 2.** If possible, integrate Eq. (4) term by term one or more times. This will reduce the order of Eq. (4). For simplicity, the integration constant(s) can be set to zero.

**step 3.** Introduce the solution $U(\zeta)$ of Eq. (4) in the finite series form

$$U(\zeta) = \sum_{i=0}^{N} a_i \left( \frac{G'(\zeta)}{G(\zeta)} \right)^i,$$

(5)

where $a_i$ are real constants with $a_N \neq 0$ to be determined, also $N$ is a positive integer to be determined. The function $G(\zeta)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(\zeta) + \lambda G'(\zeta) + \mu G(\zeta) = 0,$$

(6)

where $\lambda$ and $\mu$ are real constants to be determined.

**step 4.** Determining $N$, can be accomplished by balancing the linear term(s) of highest order derivatives with the highest order nonlinear term(s) in Eq. (4).

**step 5.** Substituting (6) together with (5) into Eq. (4) yields an algebraic equation involving powers of $\left(\frac{G'}{G}\right)$. Equating the coefficients of each power of $\left(\frac{G'}{G}\right)$ to zero gives a system of algebraic equations for $a_i$, $\lambda$, $\mu$ and $c$. Then, we solve the system with the aid of a computer algebra system (CAS), such as Maple, to determine these constants. Next, depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, we get solutions of Eq. (6). So, we can obtain exact solutions of the given Eq. (3).

### 3 Application of $\left(\frac{G'}{G}\right)$-expansion method to $(1 + 1)$-dimensional K-G-S equations

According to equations (1)-(2), $(1 + 1)$-dimensional K–G–S equations are as follows

$$i\psi_t + \frac{1}{2} \phi_{xx} + \phi \psi = 0,$$

(7)

$$\phi_{tt} - \phi_{xx} + M^2 \phi - |\psi|^2 = 0,$$

(8)

where $M$ is a constant number. Since $\psi$ is a complex function, we assume that

$$\psi = e^{i\eta} u(x,t), \quad \eta = \alpha x + \beta t,$$

(9)

where $\alpha$, $\beta$ are some constants to be determined. Substituting (9) into Eqs. (7)-(8) and canceling $e^{i\eta}$, yields

$$u_t + \alpha u_x = 0$$

$$u_{xx} - (\alpha^2 + 2\beta) u + 2\phi u = 0$$

$$\phi_{tt} - \phi_{xx} + M^2 \phi - u^2 = 0.$$

(10)

The following travelling wave variables

$$u(x,t) = U(\xi), \quad \phi(x,t) = \Phi(\xi), \quad \xi = kx + wt$$

(11)

permit us to convert equations (10) into the following ordinary differential equation

$$w U'' + k\alpha U' = 0 \Rightarrow w = -k\alpha$$

$$k^2 U'' - (\alpha^2 + 2\beta) U + 2\phi U = 0$$

$$(w^2 - k^2) \Phi'' + M^2 \Phi - U^2 = 0.$$

(12)

(13)
where the prime denotes the derivative with respect to \( \xi \). According to step 4, considering the homogeneous balance between highest order derivatives and non-linear terms in Eqs. (12)-(13) gives the leading orders \( N_1 = 2 \) for \( U \) and \( N_2 = 2 \) for \( \Phi \). Therefore, in view of the \( \left( \frac{G'}{G} \right) \)-expansion method, we can assume that the solution of Eqs. (12)-(13) is of the form

\[
U(\xi) = a_0 + a_1 \left( \frac{G'}{G} \right) + a_2 \left( \frac{G'}{G} \right)^2, \quad a_2 \neq 0,
\]

\[
\Phi(\xi) = b_0 + b_1 \left( \frac{G'}{G} \right) + b_2 \left( \frac{G'}{G} \right)^2, \quad b_2 \neq 0,
\]

where \( G = G(\xi) \) satisfies the second linear ordinary differential equation

\[
G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0,
\]

and \( a_0, a_1, a_2, b_0, b_1, b_2, \lambda \) and \( \mu \) are constants to be determined later. By using (14), (15) and (16) it is derived that

\[
U''(\xi) = a_1 \mu + 2a_2 \mu^2 + (a_1 \lambda^2 + 2a_1 \mu + 6a_2 \lambda \mu) \left( \frac{G'}{G} \right) + (3a_1 \lambda + 4a_2 \lambda^2 + 8a_2 \mu) \left( \frac{G'}{G} \right)^2 \\
+ (2a_1 + 10a_2 \lambda) \left( \frac{G'}{G} \right)^3 + 6a_2 \left( \frac{G'}{G} \right)^4,
\]

\[
U^2(\xi) = a_0^2 + (2a_0 a_1) \left( \frac{G'}{G} \right) + (2a_0 a_2 + a_1^2) \left( \frac{G'}{G} \right)^2 + 2a_1 a_2 \left( \frac{G'}{G} \right)^3 + a_2^2 \left( \frac{G'}{G} \right)^4,
\]

\[
\Phi''(\xi) = b_1 \lambda \mu + 2b_2 \mu^2 + (b_1 \lambda^2 + 2b_1 \mu + 6b_2 \lambda \mu) \left( \frac{G'}{G} \right) + (3b_1 \lambda + 4b_2 \lambda^2 + 8b_2 \mu) \left( \frac{G'}{G} \right)^2 \\
+ (2b_1 + 10b_2 \lambda) \left( \frac{G'}{G} \right)^3 + 6b_2 \left( \frac{G'}{G} \right)^4.
\]

By substituting (17)-(19) into Eqs. (12)-(13) and collecting all terms with the same power of \( \left( \frac{G'}{G} \right) \) together, the left-hand sides of Eqs. (12)-(13) are converted into two polynomials in \( \left( \frac{G'}{G} \right) \). Equating the coefficients of the polynomials to zero, yields a set of simultaneous algebraic equations for \( a_0, a_1, a_2, b_0, b_1, b_2, \alpha, \beta, k, w, \lambda \) and \( \mu \) as follows:

\[
\left( \frac{G'}{G} \right)^0 : -k^2 a_1 \lambda \mu - \alpha^2 a_0 - 2\beta a_0 + 2a_0 b_0 + 2k^2 a_2 \mu^2 = 0,
\]

\[
\left( \frac{G'}{G} \right)^1 : -\alpha^2 a_1 - 2\beta a_1 + k^2 a_1 \lambda^2 + 2k^2 a_1 \mu + 6k^2 a_2 \lambda \mu + 2a_1 b_0 + 2a_0 b_1 = 0,
\]

\[
\left( \frac{G'}{G} \right)^2 : -\alpha^2 a_2 - 2\beta a_2 + 3k^2 a_1 \lambda + 4k^2 a_2 \lambda^2 + 8k^2 a_2 \mu + 2a_1 b_1 + 2a_2 b_0 + 2a_0 b_2 = 0,
\]

\[
\left( \frac{G'}{G} \right)^3 : 2k^2 a_1 + 10k^2 a_2 \lambda + 2a_1 b_2 + 2a_2 b_1 = 0,
\]

\[
\left( \frac{G'}{G} \right)^4 : 6k^2 a_2 + 2a_2 b_2 = 0,
\]

\[
\left( \frac{G'}{G} \right)^0 : \mu^2 b_0 - \alpha_0^2 - b_1 \lambda \mu k^2 + 2b_2 \mu^2 \alpha^2 k^2 - 2b_2 \mu^2 k^2 + b_1 \lambda \mu a^2 k^2 = 0,
\]

\[
\left( \frac{G'}{G} \right)^1 : M b_1 - 2a_0 a_1 + b_1 \lambda^2 \alpha^2 k^2 - b_1 \lambda^2 k^2 + 2b_1 \mu a^2 k^2 - 2b_1 \mu k^2 + 6b_2 \lambda \mu a^2 k^2 - 6b_2 \lambda \mu k^2 = 0,
\]

\[
\left( \frac{G'}{G} \right)^2 : -\alpha_1^2 + \mu^2 b_2 - 2a_2 a_0 + 3b_1 \lambda \alpha^2 k^2 - 3b_1 \lambda k^2 + 4b_2 \lambda \alpha^2 k^2 - 4b_2 \lambda k^2 + 8b_2 \mu a^2 k^2 - 8b_2 \mu k^2 = 0,
\]

\[
\left( \frac{G'}{G} \right)^3 : -2b_1 k^2 - 2a_1 a_2 + 2b_1 \alpha^2 k^2 + 10b_2 \lambda \alpha^2 k^2 - 10b_2 \lambda k^2 = 0,
\]

\[
\left( \frac{G'}{G} \right)^4 : -\alpha_2^2 - 6b_2 k^2 + 6b_2 \alpha^2 k^2 = 0.
\]

Through solving the above equations by MAPLE, we get two different cases depending on the sign of \( 1 - \alpha^2 \).

**Case 1:** \( 1 - \alpha^2 > 0 \)

**Case 2:** \( 1 - \alpha^2 < 0 \)
First solution set:
\[ a_0 = \pm \frac{3(M^2 - k^2\lambda^2 + k^2\alpha^2\lambda^2)}{2\sqrt{2 - 2\alpha^2}}, \quad a_1 = \mp 3k^2\lambda\sqrt{2 - 2\alpha^2}, \quad a_2 = \mp 3k^2\sqrt{2 - 2\alpha^2}, \]
\[ b_0 = -\frac{3(M^2 - k^2\lambda^2 + k^2\alpha^2\lambda^2)}{4(-1 + \alpha^2)}, \quad b_1 = -3k^2\lambda, \quad b_2 = -3k^2, \]
\[ \mu = \frac{M^2 - k^2\lambda^2 + k^2\alpha^2\lambda^2}{4k^2(-1 + \alpha^2)}, \quad \beta = -\frac{\alpha^4 - \alpha^2 + M^2}{2(-1 + \alpha^2)}, \]

where \( M, k, \alpha \) and \( \lambda \) are arbitrary constants and \( \alpha \neq \pm 1, k \neq 0 \).

Substituting the solution set (20) and the corresponding solution of (16) into (14)-(15), we have three types of travelling wave solutions of Eqs. (12)-(13) as follows:

When \( \Delta = \lambda^2 - 4\mu = \frac{M^2}{k^2(1-\alpha^2)} > 0 \), we obtain hyperbolic function solutions
\[ \Phi_1(\xi) = \frac{3M^2}{4(1-\alpha^2)} H(\xi), \quad U_1(\xi) = \pm \sqrt{2 - 2\alpha^2} \Phi_1(\xi), \]

where
\[ H(\xi) = \frac{(-C_2^2 + C_1^2)}{(C_2\sinh(\frac{1}{2}\sqrt{\Delta}\xi) + C_1\cosh(\frac{1}{2}\sqrt{\Delta}\xi))^2}. \]

Considering the Eqs. (9) and (11), the corresponding generalized wave solution of \((1+1)\)-dimensional K-G-S equations (12)-(13) in this case is written as:
\[ \psi_1(\xi) = U_1(\xi)e^{i\eta}, \]
where \( \xi = (x-\alpha t)k, \eta = \alpha x - \frac{1}{2}\left[\frac{\alpha^4 - \alpha^2 + M^2}{-1 + \alpha^2}\right]t, C_1 \) and \( C_2 \) are arbitrary constants.

It should be also noted that if we set \( k = 1, \ C_1 = 1, \ C_2 = 0 \), then the following solution can be derived:
\[ \psi_1(\xi) = \mp \frac{3\sqrt{2}}{4} \frac{M^2}{\sqrt{1-\alpha^2}} \text{sech}^2\left(\frac{1}{2}\frac{M}{\sqrt{1-\alpha^2}}(x-\alpha t)\right) e^{i\left(\alpha x + \left(\frac{M^2}{2(-1 + \alpha^2)} - \frac{\alpha^2}{2}\right)t\right)}, \]

\[ \Phi_1(\xi) = \frac{3M^2}{4\sqrt{1-\alpha^2}} \text{sech}^2\left(\frac{1}{2}\frac{M}{\sqrt{1-\alpha^2}}(x-\alpha t)\right). \]

This results coincides with the solutions (31)-(32) shown in [1], (2.34)-(2.35) in [15] and (33)-(34) in [3].

When \( \Delta = \lambda^2 - 4\mu = \frac{M^2}{k^2(1-\alpha^2)} = 0 \), \( M = 0 \), we obtain the rational function solutions
\[ \Phi_2(\xi) = \frac{-3C_2^2k^2}{(C_1 + C_2\xi)^2}, \quad U_2(\xi) = \pm \sqrt{2 - 2\alpha^2} \Phi_2(\xi), \quad \psi_2(\xi) = U_2(\xi)e^{i\eta}, \]
where \( \xi = (x-\alpha t)k, \eta = \alpha x - \frac{1}{2}\left[\frac{\alpha^4 - \alpha^2 + M^2}{-1 + \alpha^2}\right]t, C_1 \) and \( C_2 \) are arbitrary constants.

Second solution set:
\[ a_0 = \pm \frac{3k^2\lambda\alpha^2 - 3k^2\lambda^2 - M^2}{2\sqrt{2 - 2\alpha^2}}, \quad a_1 = \mp 3k^2\lambda\sqrt{2 - 2\alpha^2}, \quad a_2 = \mp 3k^2\sqrt{2 - 2\alpha^2}, \]
\[ b_0 = -\frac{3k^2\lambda\alpha^2 - 3k^2\lambda^2 - M^2}{4(-1 + \alpha^2)}, \quad b_1 = -3k^2\lambda, \quad b_2 = -3k^2, \]
\[ \mu = \frac{-M^2 - k^2\lambda^2 + k^2\alpha^2\lambda^2}{4k^2(-1 + \alpha^2)}, \quad \beta = -\frac{\alpha^4 - \alpha^2 + M^2}{2(-1 + \alpha^2)}, \]

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where $M, k, \alpha$ and $\lambda$ are arbitrary constants and $\alpha \neq \pm 1, k \neq 0$.

When $\Delta = \lambda^2 - 4\mu = \frac{M^2}{k^2(-1 + \alpha^2)} < 0$, we obtain the trigonometric function solutions

$$
\Phi_3(\xi) = \frac{M^2H(\xi)}{4(1 - \alpha^2)}, \quad U_3(\xi) = \pm \sqrt{2 - 2\alpha^2}\Phi_3(\xi), \quad \psi_3(\xi) = U_3(\xi)e^{i\eta},
$$

where

$$
H(\xi) = \frac{2(C_2^2 - C_1^2)\cos\left(\frac{1}{2}\sqrt{-\Delta}\xi\right) + C_2^2 - 4C_1C_2\sin\left(\frac{1}{2}\sqrt{-\Delta}\xi\right) \cos\left(\frac{1}{2}\sqrt{-\Delta}\xi\right) + 3C_1^2}{(C_2^2 - C_1^2)\cos^2\left(\frac{1}{2}\sqrt{-\Delta}\xi\right) - C_2^2 - 2C_1C_2\sin\left(\frac{1}{2}\sqrt{-\Delta}\xi\right) \cos\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)}.}
$$

When $\lambda^2 - 4\mu = \frac{M^2}{k^2(1 - \alpha^2)} = 0, (M = 0)$, we obtain the rational function solutions

$$
\Phi_4(\xi) = \frac{-3C_2^2k^2}{(C_1 + C_2\xi)^2}, \quad U_4(\xi) = \pm \sqrt{2 - 2\alpha^2}\Phi_4(\xi), \quad \psi_4(\xi) = U_4(\xi)e^{i\eta},
$$

where $\xi = (x - \alpha t)k$, $\eta = ax - \frac{1}{2}\frac{\alpha^4 - \alpha^2 + M^2}{-1 + \alpha^2}t$, $C_1$ and $C_2$ are arbitrary constants.

**Third solution set:**

$$
a_0 = 0, \quad a_1 = \pm \frac{6M^2}{\sqrt{2 - 2\alpha^2} \lambda}, \quad a_2 = \pm \frac{6M^2}{\sqrt{2 - 2\alpha^2} \lambda^2},
$$

$$
b_0 = 0, \quad b_1 = \frac{3M^2}{\lambda(1 + \alpha^2)}, \quad b_2 = \frac{3M^2}{\lambda^2(1 + \alpha^2)},
$$

$$
\mu = 0, \quad \beta = -\frac{\alpha^4 - \alpha^2 + M^2}{2(1 + \alpha^2)}, \quad k = \pm \frac{M}{\sqrt{1 - \alpha^2} \lambda},
$$

(22)

where $M$, $\alpha$ and $\lambda$ are arbitrary constants and $\alpha \neq \pm 1, \lambda \neq 0$.

When $\Delta = \lambda^2 - 4\mu = \lambda^2 > 0, (\lambda > 0)$, we obtain the hyperbolic function solutions

$$
\Phi_5(\xi) = \frac{3M^2}{4(-1 + \alpha^2)}H(\xi), \quad U_5(\xi) = \mp \sqrt{2 - 2\alpha^2}\Phi_5(\xi), \quad \psi_5(\xi) = U_5(\xi)e^{i\eta},
$$

where

$$
H(\xi) = \frac{C_2^2 - C_1^2}{\left(C_2\sinh\left(\frac{\lambda\xi}{2}\right) + C_1\cosh\left(\frac{\lambda\xi}{2}\right)\right)^2}.
$$

When $\lambda^2 - 4\mu = \lambda^2 > 0, (\lambda < 0)$, we obtain the hyperbolic function solutions

$$
\Phi_6(\xi) = \frac{3M^2}{4(-1 + \alpha^2)}H(\xi), \quad U_6(\xi) = \mp \sqrt{2 - 2\alpha^2}\Phi_6(\xi), \quad \psi_6(\xi) = U_6(\xi)e^{i\eta},
$$

where

$$
H(\xi) = \frac{C_2^2 - C_1^2}{\left(C_2\sinh\left(\frac{\lambda\xi}{2}\right) - C_1\cosh\left(\frac{\lambda\xi}{2}\right)\right)^2}.
$$

where $\xi = (x - \alpha t)k = \pm \frac{M}{\sqrt{1 - \alpha^2} \lambda}(x - \alpha t), \eta = ax - \frac{1}{2}\frac{\alpha^4 - \alpha^2 + M^2}{-1 + \alpha^2}t$, $C_1$ and $C_2$ are arbitrary constants.
Fourth solution set:

\[ a_0 = 0, \quad a_1 = \pm \frac{2M^2}{\sqrt{2 - 2\alpha^2} \lambda}, \quad a_2 = \pm \frac{2M^2}{\sqrt{2 - 2\alpha^2} \lambda^2}, \]
\[ b_0 = 0, \quad b_1 = \frac{-M^2}{\lambda(-1 + \alpha^2)}, \quad b_2 = \frac{-M^2}{\lambda^2(-1 + \alpha^2)}, \]
\[ \mu = \frac{-\lambda^2}{2}, \quad \beta = \frac{-\alpha^4 - \alpha^2 + M^2}{2(-1 + \alpha^2)}, \quad k = \pm \frac{M}{\sqrt{-3 + 3\alpha^2} \lambda}. \]

where \( M, \alpha \) and \( \lambda \) are arbitrary constants and \( \alpha \neq \pm 1, \lambda \neq 0 \).

When \( \Delta = \lambda^2 - 4\mu = 3\lambda^2 > 0, (\lambda > 0) \), we obtain the hyperbolic function solutions

\[ \Phi_7(\xi) = \frac{-M^2}{4(-1 + \alpha^2)} H(\xi), \quad U_7(\xi) = \mp \sqrt{2 - 2\alpha^2} \Phi_7(\xi), \quad \psi_7(\xi) = U_7(\xi)e^{i\eta}, \]

where

\[ H(\xi) = \frac{2 \left( C_1 \sinh\left( \frac{\sqrt{3}}{2} \lambda \xi \right) + C_2 \cosh\left( \frac{\sqrt{3}}{2} \lambda \xi \right) \right)^2 + \left( C_2^2 - C_1^2 \right)}{\left( C_2 \sinh\left( \frac{\sqrt{3}}{2} \lambda \xi \right) + C_1 \cosh\left( \frac{\sqrt{3}}{2} \lambda \xi \right) \right)^2}. \]

When \( \Delta = \lambda^2 - 4\mu = 3\lambda^2 > 0, (\lambda < 0) \), we obtain the hyperbolic function solutions

\[ \Phi_8(\xi) = \frac{-M^2}{4(-1 + \alpha^2)} H(\xi), \quad U_8(\xi) = \mp \sqrt{2 - 2\alpha^2} \Phi_8(\xi), \quad \psi_8(\xi) = U_8(\xi)e^{i\eta}, \]

where

\[ H(\xi) = \frac{2 \left( C_1 \sinh\left( \frac{\sqrt{3}}{2} \lambda \xi \right) - C_2 \cosh\left( \frac{\sqrt{3}}{2} \lambda \xi \right) \right)^2 + \left( C_2^2 - C_1^2 \right)}{\left( C_2 \sinh\left( \frac{\sqrt{3}}{2} \lambda \xi \right) - C_1 \cosh\left( \frac{\sqrt{3}}{2} \lambda \xi \right) \right)^2}. \]

where \( \xi = (x - \alpha t)k = \mp \frac{M}{\sqrt{-3 + 3\alpha^2} \lambda}(x - \alpha t), \eta = \alpha x - \frac{1}{2} \frac{\alpha^4 - \alpha^2 + M^2}{-1 + \alpha^2} t. \)

Case 2: \( 1 - \alpha^2 < 0 \)

Fifth solution set:

\[ a_0 = \mp i \frac{3(M^2 - k^2\lambda^2 + k^2\alpha^2\lambda^2)}{2\sqrt{2\alpha^2 - 2}}, \quad a_1 = \mp i 3k^2 \lambda \sqrt{2\alpha^2 - 2}, \quad a_2 = \mp i 3k^2 \sqrt{2\alpha^2 - 2}, \]
\[ b_0 = - \frac{3(M^2 - k^2\lambda^2 + k^2\alpha^2\lambda^2)}{4(-1 + \alpha^2)}, \quad b_1 = - 3k^2 \lambda, \quad b_2 = - 3k^2, \]
\[ \mu = \frac{M^2 - k^2\lambda^2 + k^2\alpha^2\lambda^2}{4k^2(-1 + \alpha^2)}, \quad \beta = \frac{-\alpha^4 - \alpha^2 + M^2}{2(-1 + \alpha^2)}, \]

where \( M, k, \alpha \) and \( \lambda \) are arbitrary constants and \( \alpha \neq \pm 1, k \neq 0 \).

When \( \Delta = \lambda^2 - 4\mu = \frac{-M^2}{k^2(-1 + \alpha^2)} < 0 \), we obtain the trigonometric function solutions

\[ \Phi_9(\xi) = \frac{3M^2}{4(-1 + \alpha^2)} H(\xi), \quad U_9(\xi) = \mp i \sqrt{2\alpha^2 - 2} \Phi_9(\xi), \quad \psi_9(\xi) = U_9(\xi)e^{i\eta}, \]
where
\[
H(\xi) = \frac{(C_2^2 + C_1^2)}{(-C_2^2 + C_2^2)\cos^2(\frac{1}{2}\sqrt{\Delta} \xi) - 2C_1C_2\cos(\frac{1}{2}\sqrt{\Delta} \xi)\sin(\frac{1}{2}\sqrt{\Delta} \xi) - C_2^2)}.
\]

When \( \Delta = \lambda^2 - 4\mu = \frac{M^2}{k^2(1 - \alpha^2)} = 0 \), \((M = 0)\), we obtain the rational functions
\[
\Phi_{10}(\xi) = \frac{-3\epsilon_2^2\lambda^2}{(c_1 + \epsilon_2^2)^2}, \quad U_{10}(\xi) = \mp i\sqrt{2\alpha^2 - 2}\Phi_{10}(\xi), \quad \psi_{10}(\xi) = U_{10}(\xi)e^{i\eta},
\]
where \( \xi = (x - \alpha t)k, \eta = ax - \frac{\alpha^4 - \alpha^2 + M^2}{2\epsilon - 1 + \alpha^2}t \), \(C_1\) and \(C_2\) are arbitrary constants.

Sixth solution set:
\[
a_0 = \mp i\frac{3\epsilon_2^2\lambda^2 - 3\epsilon_2^2\lambda^2 - M^2}{2\sqrt{2\alpha^2 - 2}}, \quad a_1 = \mp 3\epsilon_2^2i\lambda\sqrt{2\alpha^2 - 2}, \quad a_2 = \mp 3\epsilon_2^2i\sqrt{2\alpha^2 - 2},
\]
\[
b_0 = -\frac{3\epsilon_2^2\lambda^2 - 3\epsilon_2^2\lambda^2 - M^2}{4(-1 + \alpha^2)}, \quad b_1 = -3\epsilon_2^2\lambda, \quad b_2 = -3\epsilon_2^2.
\]
\[
\mu = \frac{-M^2 - k^2\lambda^2 + k^2\alpha^2\lambda^2}{4k^2(-1 + \alpha^2)}, \quad \beta = -\frac{\alpha^4 - \alpha^2 + M^2}{2(-1 + \alpha^2)},
\]
where \(M, k, \alpha\) and \(\lambda\) are arbitrary constants and \(\alpha \neq \pm 1, k \neq 0\).

When \( \Delta = \lambda^2 - 4\mu = \frac{M^2}{k^2(1 - \alpha^2)} > 0 \), \((M = 0)\), we obtain the hyperbolic functions
\[
\Phi_{11}(\xi) = -\frac{M^2}{4(-1 + \alpha^2)}H(\xi), \quad U_{11}(\xi) = \mp i\sqrt{2\alpha^2 - 2}\Phi_{11}(\xi), \quad \psi_{11}(\xi) = U_{11}(\xi)e^{i\eta},
\]
where
\[
H(\xi) = \frac{2(C_2^2 + C_1^2)\cosh^2(\frac{1}{2}\sqrt{\Delta} \xi) + C_2^2 + 4C_1C_2\sinh(\frac{1}{2}\sqrt{\Delta} \xi)\cosh(\frac{1}{2}\sqrt{\Delta} \xi) - 3C_1^2}{C_2\sinh(\frac{1}{2}\sqrt{\Delta} \xi) + C_1\cosh(\frac{1}{2}\sqrt{\Delta} \xi)}.
\]

When \( \Delta = \lambda^2 - 4\mu = \frac{M^2}{k^2(1 - \alpha^2)} = 0 \), \((M = 0)\), we obtain the rational functions
\[
\Phi_{12}(\xi) = \frac{-3\epsilon_2^2\lambda^2}{(c_1 + \epsilon_2^2)^2}, \quad U_{12}(\xi) = \mp i\sqrt{2\alpha^2 - 2}\Phi_{12}(\xi), \quad \psi_{12}(\xi) = U_{12}(\xi)e^{i\eta},
\]
where \( \xi = (x - \alpha t)k, \eta = ax - \frac{\alpha^4 - \alpha^2 + M^2}{2\epsilon - 1 + \alpha^2}t \), \(C_1\) and \(C_2\) are arbitrary constants.

Seventh solution set:
\[
a_0 = 0, \quad a_1 = \mp i\frac{6M^2}{\sqrt{2\alpha^2 - 2}\lambda}, \quad a_2 = \mp i\frac{6M^2}{\sqrt{2\alpha^2 - 2}\lambda^2},
\]
\[
b_0 = 0, \quad b_1 = \frac{3M^2}{\lambda(-1 + \alpha^2)}, \quad b_2 = \frac{3M^2}{\lambda^2(-1 + \alpha^2)},
\]
\[
\mu = 0, \quad \beta = -\frac{\alpha^4 - \alpha^2 + M^2}{2(-1 + \alpha^2)}t, \quad k = \mp i\frac{M}{\sqrt{\alpha^2 - 1}\lambda},
\]
where \(M, \alpha\) and \(\lambda\) are arbitrary constants and \(\alpha \neq \pm 1, \lambda \neq 0\).
When $\Delta = \lambda^2 - 4\mu = \lambda^2 > 0$, $(\lambda > 0)$, we obtain the hyperbolic function solutions

$$
\Phi_{13}(\xi) = \frac{3M^2}{4(-1 + \alpha^2)} H(\xi), \quad U_{13}(\xi) = \mp i\sqrt{2\alpha^2 - 2} \Phi_{13}(\xi), \quad \psi_{13}(\xi) = U_{13}(\xi)e^{in},
$$

where

$$
H(\xi) = \frac{C_2^2 - C_1^2}{\left( C_2\sinh\left(\frac{\lambda\xi}{2}\right) + C_1\cosh\left(\frac{\lambda\xi}{2}\right) \right)^2}.
$$

When $\Delta = \lambda^2 - 4\mu = \lambda^2 > 0$, $(\lambda < 0)$, we obtain the hyperbolic function solutions

$$
\Phi_{14}(\xi) = \frac{3M^2}{4(-1 + \alpha^2)} H(\xi), \quad U_{14}(\xi) = \mp i\sqrt{2\alpha^2 - 2} \Phi_{14}(\xi), \quad \psi_{14}(\xi) = U_{14}(\xi)e^{in},
$$

where

$$
H(\xi) = \frac{C_2^2 - C_1^2}{\left( C_2\sinh\left(\frac{\lambda\xi}{2}\right) - C_1\cosh\left(\frac{\lambda\xi}{2}\right) \right)^2}.
$$

where $\xi = (x - \alpha t)k = \mp i\frac{M}{\sqrt{\alpha^2 - 1}}(x - \alpha t), \eta = \alpha x - \frac{1}{2} \alpha^4 - \alpha^2 + M^2 - t$, $C_1$ and $C_2$ are arbitrary constants.

**Eighth solution set:***

$$
a_0 = 0, \quad a_1 = \mp i\frac{2M^2}{\sqrt{2\alpha^2 - 2} \lambda}, \quad a_2 = \mp i\frac{2M^2}{\sqrt{2\alpha^2 - 2} \lambda^2},
$$

$$
b_0 = 0, \quad b_1 = -\frac{M^2}{\lambda(-1 + \alpha^2)}, \quad b_2 = -\frac{M^2}{\lambda^2(-1 + \alpha^2)},
$$

$$
\mu = -\frac{\lambda^2}{2}, \quad \beta = -\frac{\alpha^4 - \alpha^2 + M^2}{2(-1 + \alpha^2)}, \quad k = \pm i\frac{M}{\sqrt{3\alpha^2 - 3}},
$$

where $M, \alpha$ and $\lambda$ are arbitrary constants and $\alpha \neq \pm 1, \lambda \neq 0$.

When $\Delta = \lambda^2 - 4\mu = 3\lambda^2 > 0$, $(\lambda > 0)$, we obtain the hyperbolic function wave solutions

$$
\Phi_{15}(\xi) = \frac{-M^2}{4(-1 + \alpha^2)} H(\xi), \quad U_{15}(\xi) = \pm i\sqrt{2\alpha^2 - 2} \Phi_{15}(\xi), \quad \psi_{15}(\xi) = U_{15}(\xi)e^{in},
$$

where

$$
H(\xi) = \frac{2 \left( C_1\sinh\left(\frac{\lambda\xi}{2}\right) + C_2\cosh\left(\frac{\lambda\xi}{2}\right) \right)^2 + \left( C_2^2 - C_1^2 \right)}{\left( C_2\sinh\left(\frac{\lambda\xi}{2}\right) + C_1\cosh\left(\frac{\lambda\xi}{2}\right) \right)^2}.
$$

When $\Delta = \lambda^2 - 4\mu = 3\lambda^2 > 0$, $(\lambda < 0)$, we obtain the hyperbolic function solutions

$$
\Phi_{16}(\xi) = \frac{-M^2}{4(-1 + \alpha^2)} H(\xi), \quad U_{16}(\xi) = \pm i\sqrt{2\alpha^2 - 2} \Phi_{16}(\xi),
$$

where

$$
H(\xi) = \frac{2 \left( C_1\sinh\left(\frac{\lambda\xi}{2}\right) - C_2\cosh\left(\frac{\lambda\xi}{2}\right) \right)^2 + \left( C_2^2 - C_1^2 \right)}{\left( C_2\sinh\left(\frac{\lambda\xi}{2}\right) - C_1\cosh\left(\frac{\lambda\xi}{2}\right) \right)^2}.
$$

where $\xi = (x - \alpha t)k = \mp i\frac{M}{\sqrt{3 + 3\alpha^2}}(x - \alpha t), \eta = \alpha x - \frac{1}{2} \alpha^4 - \alpha^2 + M^2 - t$. 

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4 Conclusions

In this paper, we have seen that three types of traveling wave complex solutions in terms of hyperbolic, trigonometric and rational functions for the \((1 + 1)\)-dimensional K-G-S equations are successfully found out by using the \((\frac{G'}{G})\)-expansion method. We have found some complex solutions for the K-G-S equations that are very difficult to be solved by traditional methods. The performance of this method is reliable, simple and gives many exact solutions. As a result, a series of some exact complex solutions are obtained and the solutions given in [1, 3, 15] are derived as special cases. Solutions presented in this paper have been checked with Maple by putting them back into the Eqs. (12)-(13).

References