A Method for Recovering the Shape for Inverse Scattering Problem of Acoustic Waves

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Abstract: The inverse problem considered in this paper is to determine the shape of an obstacle from a knowledge of the time-harmonic incident field and the phase and amplitude of the far field pattern of the scattered wave in two-dimension. Single- and double-layer potential are combined to approach the scattered waves. An approximate method containing a hyper-singular operator is presented and the convergence of this method is proven. Numerical examples are given to show that this method is both accurate and simple to use.

Keywords: Impedance boundary condition; Helmholtz equation; Hyper-singular operator

1 Introduction

The inverse scattering problem for time-harmonic acoustic waves in two-dimension has been considered for Dirichlet and impedance boundary condition in a series of papers[1-4]. All these papers give a full treatment of the corresponding problems including the convergence and the numerical solution but not considering the hyper-singular operator $T$. In this paper, it provide a method to the case of the impedance boundary condition for acoustic waves, which includes the hyper-singular operator. In comparison with [3], the combined single-layer and double-layer potential are used to approach the scattering waves. The shape is determined by the given far field pattern of the scattered waves. In this method, the numerical method of the hyper-singular operator is provided and the convergence is proceeded. Numerical examples are given to show that this method is more accurate than [3].

2 Mathematical analysis of the inverse scattering problem

Let $D$ be a bounded, connected domain in the plane with boundary $\partial D \in C^2$ and let the incident field $u^i$ be given by $u^i(x) = \exp[i k x \cdot d]$ where $k > 0$ is the wave number and $d$ is a fixed unit vector. If we denote the scattering field by $u^s$ and define the total field $u$ by $u = u^i + u^s$, then the direct scattering problem for an acoustic obstacle $D$ is to find a solution $u \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C(\mathbb{R}^2 \setminus D)$ of the Helmholtz equation

$$\Delta_2 u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus D,$$

which satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} + ik \lambda(x) u = 0 \quad \text{on } \partial D,$$

and $u^s$ satisfies the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left\{ \frac{\partial u^s}{\partial r} - ik u^s \right\} = 0, \quad r = |x|,$$

uniformly in all directions $x/|x|$.

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Thus, we get the approach of the scattered wave \( u^s \). From Green’s formula and the asymptotic behavior of the Hankel function \( H^{(1)}_0 \), we can easily show\(^5\) that

\[
\begin{align*}
  u_\infty(x) & = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \left\{ u^s(y) \frac{\partial e^{-ik\hat{x} \cdot y}}{\partial y} - \frac{\partial u^s}{\partial \nu}(y)e^{-ik\hat{x} \cdot y} \right\} ds(y), \\
  & \quad \hat{x} \in \Omega,
\end{align*}
\]

for \( |\hat{x}| = x/|x| \).

For the problem (1)-(3), there exists the following theorem.

**Theorem 2.1 (6)** The exterior impedance boundary-value problem has at most one solution provided \( \text{Im}(k\lambda) \geq 0 \) on \( \partial D \). The solution \( u^s \) in \( \mathbb{R}^2 \setminus D \) and each differentiation of \( u^s \) in \( \mathbb{R}^2 \setminus \bar{D} \) depend continuously on the boundary data.

Let \( \Gamma \in C^2 \) be closed curve contained in \( D \) and \( k^2 \) is not a Dirichlet eigenvalue of Laplacian in \( \Gamma \). Let the combined single- and double-layer potential

\[
v(x) = \int_\Gamma \varphi(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y) - i\eta \int_\Gamma \varphi(y) \Phi(x, y) ds(y), \quad \varphi \in L^2(\Gamma)
\]

approach the scattered field \( u^s \), where \( \Phi(x, y) = \frac{1}{4\pi} H^{(1)}_0(k|x - y|) \) denotes the fundamental solution to the Helmholtz equation in two-dimension. From \( v(x) \) and \( H^{(1)}_0 \), we see that the far-field pattern of the combined potential (6) is given by

\[
u_{\infty}(\hat{x}) = \frac{e^{-i\pi/4}}{\sqrt{8\pi k}} \int_{\Gamma} \{ k\hat{x} \cdot \nu(y) + \eta \} e^{-ik\hat{x} \cdot y} \varphi(y) ds(y).
\]

Hence, for the given far-field pattern, we should solve the integral equation

\[
(F\varphi)(\hat{x}) = u_{\infty}(\hat{x}),
\]

where \( F : L^2(\Gamma) \rightarrow L^2(\Omega) \) defined by the form

\[
(F\varphi)(\hat{x}) := \frac{e^{-i\pi/4}}{\sqrt{8\pi k}} \int_{\Gamma} \{ k\hat{x} \cdot \nu(y) + \eta \} e^{-ik\hat{x} \cdot y} \varphi(y) ds(y), \quad \hat{x} \in \Omega.
\]

The Eq.(8) is strong ill-posed problem, so we use the Tikhonov regularization method to solve this problem, that is for the regularization parameter \( \alpha > 0 \), find the solution \( \varphi_\alpha \in L^2(\Gamma) \) satisfying

\[
\| F\varphi_\alpha - u_{\infty} \|_{L^2(\Omega)}^2 + \alpha \| \varphi_\alpha \|_{L^2(\Gamma)}^2 \leq \inf_{\varphi \in L^2(\Gamma)} \left\{ \| F\varphi - u_{\infty} \|_{L^2(\Omega)}^2 + \alpha \| \varphi \|_{L^2(\Gamma)}^2 \right\}.
\]

Thus, we get the approach of the scattered wave

\[
u_{\infty} = \int_{\Gamma} \varphi_\alpha(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(y) - i\eta \int_{\Gamma} \varphi_\alpha(y) \Phi(x, y) ds(y).
\]

We should find \( \rho(\theta) \) in a set \( C^2([0, 2\pi]) \), which minimizes the impedance boundary condition

\[
\inf_{\rho(\theta) \in C^2([0, 2\pi])} \left\| \frac{\partial}{\partial \nu} (u^t(\rho(\theta)) + u_{\alpha}^s(\rho(\theta))) + ik\lambda(u^t(\rho(\theta)) + u_{\alpha}^s(\rho(\theta))) \right\|_{L^2(\partial D)}.
\]

Define operator \( G : L^2(\Gamma) \rightarrow L^2(\partial D) \)

\[
(G\varphi)(x) := (K - i\eta S)(x),
\]

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Theorem 2.2 The operator $G$, defined by (11), is injective and has dense range provided $k^2$ is not a Dirichlet eigenvalue for the negative Laplacian in the interior of $\Gamma$.

For the boundary $\rho(\theta)$, we can define the minimization problem

$$
\mu(\rho, \varphi; \alpha) = \min_{\varphi \in L^2(\Gamma)} \{ \| F\varphi - u_\infty \|_{L^2(\Gamma)}^2 + \alpha \| \varphi \|_{L^2(\Gamma)}^2 + \left\| \frac{\partial}{\partial \nu}(u^i(\rho(\theta)) + (G(\rho(\theta)))\varphi) + i\kappa\lambda(u^i(\rho(\theta)) + (G(\rho(\theta)))\varphi) \right\|_{L^2[0,2\pi]}^2 \}.
$$

(12)

3 Convergence analysis

Definition 3.1 Given the incident field $u^i$, a far field pattern $u_\infty$, and a regularization parameter $\alpha > 0$, a function $\rho_0 \in C^2[0, 2\pi]$ will be called admissible if there exists function $\varphi_0 \in L^2(\Gamma)$ such that $(\varphi_0, \rho_0)$ minimizes the expression in (12) over all $\varphi \in L^2(\Gamma)$ and $\rho \in C^2[0, 2\pi]$, that is, we have

$$
\mu(\rho_0, \varphi_0; \alpha) = m(\alpha)
$$

where

$$
m(\alpha) := \inf_{(\varphi, \rho) \in L^2(\Gamma) \times C^2[0, 2\pi]} \mu(\rho, \varphi; \alpha).
$$

Theorem 3.1 For each $\alpha > 0$, there exists an optimal solution $\rho_0 \in C^2[0, 2\pi]$.

Proof. Let $(\varphi_n, \rho_n)$ be a minimizing sequence in $L^2(\Gamma) \times C^2[0, 2\pi]$, i.e.,

$$
\lim_{n \to \infty} \mu(\varphi_n, \rho_n; \alpha) = m(\alpha).
$$

The sequence $\{\rho_n\}$ lies in a compact set $C^2[0, 2\pi]$ and hence there exists a convergent subsequence. We can assume that $\rho_n \to \rho_0, n \to \infty$. From

$$
\alpha \| \varphi_n \| \leq \mu(\varphi_n, \rho_n; \alpha) \to m(\alpha), \quad n \to \infty,
$$

and $\alpha > 0$, we conclude that the sequence $\{\varphi_n\}$ is bounded, i.e., $\| \varphi_n \|_{L^2(\Gamma)} \leq c$ for all $n$ and some constant $c$. Hence, we can assume that it converges weakly $\varphi_n \to \varphi_0 \in L^2(\Gamma)$ as $n \to \infty$.

Since $F : L^2(\Gamma) \to F : L^2(\partial D)$ and $G : L^2(\Gamma) \to F : L^2[0, 2\pi]$ represent compact operators, it follows that

$$
F\varphi_n \to F\varphi_0, G\varphi_n \to G\varphi_0, \quad n \to \infty.
$$

This now implies

$$
\alpha \| \varphi_n \|^2_{L^2(\Gamma)} = \mu(\varphi_n, \rho_n; \alpha) - \| F\varphi_n - u_\infty \|^2_{L^2(\Gamma)} - \left\| \frac{\partial}{\partial \nu}(u^i + G\varphi_n) + i\kappa\lambda(u^i + G\varphi_n) \right\|_{L^2[0,2\pi]}^2
\to m(\alpha) - \| F\varphi_0 - u_\infty \|^2_{L^2(\Gamma)} - \left\| \frac{\partial}{\partial \nu}(u^i + G\varphi_0) + i\kappa\lambda(u^i + G\varphi_0) \right\|_{L^2[0,2\pi]}^2
\leq \alpha \| \varphi_0 \|^2_{L^2(\Gamma)} \quad \text{for } n \to \infty.
$$
Since we already know weak convergence $\varphi_n \rightarrow \varphi_0$ as $n \rightarrow \infty$, it follows that

$$\lim_{n \rightarrow \infty} \|\varphi_n - \varphi_0\|_{L^2(\Gamma)} = \lim_{n \rightarrow \infty} (\|\varphi_n\|_{L^2(\Gamma)}^2 - \|\varphi_0\|_{L^2(\Gamma)}^2) \leq 0,$$

i.e., we also have convergence $\varphi_n \rightarrow \varphi_0$ as $n \rightarrow \infty$ in norm. Finally, from the continuity, we have

$$\mu(\varphi_0, \rho_0; \alpha) = \lim_{n \rightarrow \infty} \mu(\varphi_n, \rho_n; \alpha) = m(\alpha),$$

and this completes the theorem. ■

**Theorem 3.2** Let $u_\infty$ be the far field pattern corresponding to the incident field $u^i, \rho(\theta) \in C^2[0, 2\pi]$, then we have convergence of the cost functional

$$\lim_{\alpha \rightarrow 0} m(\alpha) = 0.$$  \hfill (13)

**Proof.** From theorem 2.1 and the boundary condition (2), there exists positive constant $c_1$, which makes

$$\|F\varphi - u_\infty\|_{L^2(\Omega)} \leq c_1 \left\| \frac{\partial}{\partial \nu} (G\varphi - u^s) + i k \lambda (G\varphi - u^s) \right\|_{L^\infty[0, 2\pi]}.$$  \hfill (14)

By Dirichlet to Neumann map theorem, there exists positive constant $c_2$, which makes

$$\left\| \frac{\partial}{\partial \nu} (G\varphi - u^s) + i k \lambda (G\varphi - u^s) \right\|_{C^{1, \beta}[\partial D]} \leq c_2 \|G\varphi - u^s\|_{C^{1, \beta}[0, 2\pi]}.$$  \hfill (15)

From (2), (14) and (15), there exists positive constant $c$, which makes

$$\mu(\varphi, \lambda; \alpha) \leq c \|G\varphi - u^s\|_{C^{1, \beta}[0, 2\pi]}^2 + \alpha \|\varphi\|_{L^2(\Gamma)}^2.$$  \hfill (16)

By theorem 2.2, for arbitrary $\varepsilon > 0$, there exists $\varphi \in C^{1, \beta}(\Gamma) \subset L^2(\Gamma)$, which makes

$$\|G\varphi - u^s\|_{C^{1, \beta}[0, 2\pi]} \leq \varepsilon.$$  \hfill (17)

From (16) and (17), we have

$$\alpha \leq \mu(\varphi, \lambda; \alpha) \leq c \varepsilon^2 + \alpha \|\varphi\|_{L^2(\Gamma)}^2 \rightarrow c \varepsilon^2, \quad \alpha \rightarrow 0.$$

Since $\varepsilon$ is arbitrary, (13) follows. ■

**Theorem 3.3** If it is satisfies the condition of theorem 3.2, $\alpha_n > 0, n = 1, 2, \cdots$ is a sequence converging to zero, $\{\rho_n\}$ is the admissible solutions corresponding to it, then $\rho_n(\theta) \rightarrow \rho(\theta)$ as $n \rightarrow \infty$.

**Proof.** The sequence $\{\rho_n\}$ lies in a compact set $C^2[0, 2\pi]$ and hence there exists a convergent subsequence, which we again denote by $\{\rho_n\}$, and $\rho_n \rightarrow \rho^* \in C^2[0, 2\pi]$. We want to show that $\rho^*(\theta) = \rho(\theta)$.

Let $u^*$ be the scattering waves corresponding to the boundary $\rho^*$, that is, it satisfies the boundary condition

$$\frac{\partial}{\partial \nu} (u^*(\rho^*) + u^i(\rho^*)) + i k \lambda (u^*(\rho^*) + u^i(\rho^*)) = 0, \quad \text{on} \ \partial D.$$  \hfill (18)

$\rho_n$ is the admissible solution corresponding to $\alpha_n$, and by definition 3.1, there exists $\varphi_n \in L^2(\Gamma)$ such that

$$\mu(\varphi_n, \rho_n; \alpha) = m(\alpha_n).$$

By theorem 3.2, these boundary data satisfy

$$\lim_{n \rightarrow \infty} \left\| \frac{\partial}{\partial \nu} (G(\rho_n)\varphi_n + u^i(\rho_n)) + i k \lambda G(\rho_n)\varphi_n + u^i(\rho_n) \right\|_{L^2[0, 2\pi]} = 0.$$

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Now we have
\[
\lim_{n \to \infty} \left\| \left( \frac{\partial}{\partial \nu} G(\rho_n) \varphi_n + ik\lambda G(\rho_n) \varphi_n \right) - \left( \frac{\partial}{\partial \nu} u^*(\rho^*) + ik\lambda u^*(\rho^*) \right) \right\|_{L^2[0,2\pi]} = 0.
\]

From theorem 2.1 and the boundary condition (2), the far field pattern \( F\varphi_n \) of the combined acoustic double- and single-player potential
\[
u(x) = \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \mathrm{d} s(y) - i\eta \int_{\Gamma} \varphi(y) \Phi(x,y) \mathrm{d} s(y)
\]
converges to the far field pattern \( u_\infty^* \) of \( u^* \).
On the other hand, from theorem 3.2
\[
\| F\varphi_n - u_\infty^* \|_{L^2(\Omega)} \to 0, \quad n \to \infty,
\]
we have
\[
u_\infty = u_\infty^*,
\]
that is
\[
u^* = u_\infty^*.
\]
From (2) and (18), we have
\[
\frac{\partial}{\partial \nu}(u^*(\rho) - u^*(\rho^*)) + ik\lambda(u^*(\rho) - u^*(\rho^*)) = 0, \quad \text{on } \partial D.
\]
Notice that \( u^*(\rho) = e^{ik\rho \cos(\theta - \phi)} \), where \( \phi \) denotes the incoming angle. So we have
\[
\rho(\theta) = \rho^*(\theta), \quad \theta \in [0,2\pi],
\]
and the proof is completed. \( \blacksquare \)

### 4 Numerical examples

In this section, we shall discuss the numerical implementation of the algorithm presented in the previous section. The data for the inverse problem is the far field pattern for a variety of incoming waves and choices of the wave number \( k \). In order to get the better result, the incoming waves are written as \( u^i_N(x) = \sum_{p=1}^N e^{ikx \cdot d_p} \) and \( u_\infty^N(x) \) is the far field pattern corresponding to it. For our examples, this data is generated by approximately solving the direct scattering problem (given the obstacle \( D \)) by the method introduced in [7].

In order to discretize the inverse problem (12), let the boundary curve be \( \Gamma : \{ x(\theta) = (x_1(\theta), x_2(\theta)) \} \), where \( x_1(\theta) = \rho(\theta) \cos \theta, \ x_2(\theta) = \rho(\theta) \sin \theta, \ 0 \leq \theta \leq 2\pi \). The integrals are approximated using the trapezium rule with \( \theta_j = \frac{j\pi}{N}, \ j = 0, 1, \ldots, 2n - 1 \) and \( \psi(\theta) = \varphi(y) \). By \( \frac{\partial}{\partial \nu}((K - i\eta S)\varphi) = (T - i\eta K' + inI)\varphi \), the representation (12) will be written to
\[
\mu(\psi, \rho; \alpha) = \sum_{q=0}^{L-1} \left\| e^{-i\pi/4} \frac{n}{\sqrt{8\pi k}} \sum_{j=0}^{2n-1} (k\hat{x}_q \cdot (x_2(\theta_j), -x_1(\theta_j))
\]
\[
+ \eta \| x'(\theta_j) \| e^{-ik\hat{x}_q \cdot (x_1(\theta_j), x_2(\theta_j))} \psi(\theta_j) - u_\infty(\hat{x}_q) \|^2 + \alpha \sum_{j=0}^{2n-1} \| \psi(\theta_j) \|^2
\]
\[
\quad + \left\| \sum_{j=0}^{2n-1} \sum_{p=0}^{N-1} \frac{\partial e^{ikx(\theta_j) \cdot d_p}}{\partial \nu} + (T(\rho(\theta_j)) - K'(\rho(\theta_j)) + nI) \psi(\theta_j)
\]
\[
+ ik \lambda \left( \sum_{p=0}^{N-1} e^{ikx(\theta_j) \cdot d_p} + (K(\rho(\theta_j)) - i\eta S(\rho(\theta_j))) \psi(\theta_j) \right) \right\|^2.
\]
For the parametrization of the operator $S, K, K'$ see [7], and for the hyper-singular operator $T$, we make use of the Maue's identity

$$T\varphi = \frac{d}{ds}S\frac{d\varphi}{ds} + k^2 \nu \cdot S(\nu \varphi), \quad \text{on } \Gamma.$$ 

Therefore, we can carry out a partial integration

$$\frac{d}{ds}S\frac{d\varphi}{ds}(x(\theta)) = \frac{i}{2|x'(\theta)|} \int_0^{2\pi} \frac{\partial}{\partial \theta} H_0^{(1)}(k|x(\theta) - x(\tau)|) \frac{d\varphi(x(\tau))}{d\tau} d\tau$$

$$= \frac{1}{|x'(\theta)|} \int_0^{2\pi} \left\{ \frac{\tau - \theta}{2} \frac{d\varphi(x(\tau))}{d\tau} - N(\theta, \tau) \varphi(x(\tau)) \right\} d\tau,$$

where we have set

$$\tilde{N}(\theta, \tau) = \frac{|x'(\theta)| (x(\theta) - x(\tau)) [x'(\tau) \cdot (x(\theta) - x(\tau))]}{|x(\theta) - x(\tau)|^2}.$$ 

We will set

$$N_1(\theta, \tau) = -\frac{1}{2\pi} \tilde{N}(\theta, \tau) \left\{ k^2 J_0(k|x(\theta) - x(\tau)|) - \frac{2k J_1(k|x(\theta) - x(\tau)|)}{|x(\theta) - x(\tau)|} \right\}$$

$$- \frac{k(x'(\theta) \cdot x'(\tau)) J_1(k|x(\theta) - x(\tau)|)}{2\pi |x(\theta) - x(\tau)|},$$

$$N_2(\theta, \tau) = N(\theta, \tau) - N_1(\theta, \tau) \ln(4 \sin^2 \frac{\theta - \tau}{2})$$

be analytic functions with diagonal terms

$$N_1(\theta, \theta) = -\frac{k^2 |x'(\theta)|^2}{4\pi},$$

$$N_2(\theta, \theta) = (\pi - 1 - 2c - 2 \ln k |x'(\theta)|^2 k^2 |x'(\theta)|^2 + \frac{1}{12\pi}$$

$$+ \frac{|x'(\theta) \cdot x''(\theta)|^2}{2\pi |x'(\theta)|^4} - \frac{|x'''(\theta)|^2}{4\pi |x'(\theta)|^2} - \frac{x'(\theta) \cdot x''''(\theta)}{6\pi |x'(\theta)|^2},$$

where $c = 0.57721 \cdots$ denotes Euler's constant.

For the numerical method, we will use the following interpolators quadrature rules

$$\frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\tau - \theta}{2} f'(\tau) d\tau \approx \sum_{j=0}^{2n-1} T_j^{(n)}(\theta) f(\theta_j^{(n)}),$$

$$\int_0^{2\pi} \ln(4 \sin^2 \frac{\theta - \tau}{2}) f(\tau) d\tau \approx \sum_{j=0}^{2n-1} R_j^{(n)}(\theta) f(\theta_j^{(n)}),$$

where

$$T_j^{(n)}(\theta) = -\frac{1}{n} \sum_{m=1}^{n-1} m \cos m(\theta - \theta_j^{(n)}) - \frac{1}{2} \cos n(\theta - \theta_j^{(n)}),$$

$$R_j^{(n)}(\theta) = -\frac{2\pi}{n} \sum_{m=1}^{n-1} \frac{1}{m} \cos m(\theta - \theta_j^{(n)}) - \frac{\pi}{n^2} \cos n(\theta - \theta_j^{(n)}).$$

Apply the parametrization to the operators $S, K, K', T, (19)$ will be parameterized.

In order to discretize the inverse problem (19), we approximate the functions $\varphi$ and $\rho$ by finite trigonometric series

$$\varphi_\alpha(x(\theta)) = \sum_{j=-n_1}^{n_1} g_j e^{i j \theta}, \quad g_j \in \mathbb{C},$$

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\[ \rho_\alpha(\theta) = a_0 + \sum_{j=1}^{n_2} (a_j \cos j\theta + b_j \sin j\theta), \quad a_j, b_j \in \mathbb{R}. \]

We now report on the examples we have computed. The approximate minimum of \( \mu \) occurred at \( k = 1.0, \lambda = 1.0 \) and the fixed unit vector

\[ d_p = \left( \begin{array}{c} \cos(2\pi p/3) \\ \sin(2\pi p/3) \end{array} \right), \quad (p = 0, 1, 2). \]

In our examples, the full line denotes graph of \( \rho \), and the broken line denotes graph of \( \rho_\alpha \).

![Graph of the pinched ellipse.](image1)

![Graph of the garlic.](image2)

**Example 4.1** The pinched ellipse. Exact figure: The pinched ellipse \( \rho(\theta) = 1 - \frac{1}{2} \cos 2\theta \). Parameters: \( M = 3, n = 64, n_1 = 8, n_2 = 6, L = 16, \alpha = 1.0E - 10 \). The resulting \( \rho_\alpha \) is shown in Fig.1.

**Example 4.2** The garlic. Exact figure: The garlic \( \rho(\theta) = 1 - \sin \theta \cos^2 \theta \). Parameters: \( M = 3, n = 64, n_1 = 8, n_2 = 4, L = 16, \alpha = 1.0E - 10 \). The resulting \( \rho_\alpha \) is shown in Fig.2.

### References


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