ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.8(2009) No.4,pp.504-512

Variational Iteration Technique for Solving Initial and Boundary Value Problems

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(Received 11 April 2009, accepted 3 June 2009)

Abstract: This paper outlines a detailed study of the applications of He's variational iteration method (VIM). The proposed method proves to be very efficient and effective for the solution of initial and boundary value problems having the diversified physical nature. The use of Lagrange multiplier makes the calculation easier and reduces the successive application of integral operator while still maintaining a very high level of accuracy. Several examples are given to test and re-confirm the efficiency of the suggested VIM.

Keywords: Variational iteration method; nonlinear problems; Lagrange multiplier; error estimates

1 Introduction

With the rapid development of nonlinear sciences, many analytical and numerical techniques have been developed by various scientists. Most of the developed techniques have their limitations like limited convergence, divergent results, linearization, discretization, unrealistic assumptions, calculation of complex Adomian's polynomials and non-compatibility with the physical nature of the problems [1-8]. He developed the variational iteration method (VIM) [3, 4] which proved to be fully synchronized with the versatile and complex nature of the physical problems and has been applied to a wide class of initial and boundary value problems, see [1-8] and the references therein. In this paper, we have made a comprehensive and detailed study pertaining the applications of the variational iteration method (VIM) in various diversified problems related to physics, astrophysics, astronomy, hydrodynamic and Hydromagntic stability, A-type stars, beam and plate defection theory, shallow water waves, applied and engineering sciences [4, 6]. It is concluded that the VIM is very reliable and effective for solving such problems. The use of Lagrange multiplier reduces the successive applications of integral operator, makes the solution procedure simple while still marinating a very high level of accuracy and hence give it a wider and better applicability as compare to other techniques like decomposition which involves the complexities of the Adomian's polynomials. Several examples are given to re-confirm the efficiency of the proposed variational iteration method (VIM).

2 Numerical Application

In this section, we apply the VIM for solving various initial and boundary value problems.

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IJNS.2009.12.30/309

2.1 Example

Consider the Thomas-Fermi equation

$$y''(x) = \frac{y^{3/2}}{x^{1/2}}, \quad y(0) = 1, \lim_{x\to\infty} y(x) = 0.$$

The correction functional is given as

$$y_{n+1}(x) = y_0(x) + \int_0^x \lambda(s) \left(\frac{d^2 y_n(x, s)}{ds^2} - x^{-1/2}\, \tilde{y}_n^{3/2} \right) ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = (s - x)$; the following iterative scheme is obtained:

$$y_{n+1}(x) = y_0(x) + \int_0^x (s - x) \left(\frac{d^2 y_n(x, s)}{ds^2} - x^{-1/2} y_n^{3/2} \right) ds.$$

Now, we apply a slight modification in the conventional initial value and take $y_0(x) = 1$,instead of $y_0(x) = 1 + Bx,$ where $B = y'(0)$. Consequently, following approximants are obtained

$$y_0(x) = 1,$$

$$y_1(x) = 1 + Bx + \frac{4}{3}x^{3/2},$$

$$y_2(x) = 1 + Bx + \frac{4}{3}x^{3/2} + \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3,$$

$$y_3(x) = 1 + Bx + \tfrac{4}{3}x^{3/2} + \tfrac{2}{5}Bx^{5/2} + \tfrac{1}{3}x^3 + \tfrac{3}{70}B^2x^{7/2} + \tfrac{2}{15}Bx^4 + \tfrac{2}{27}x^{9/2} + \tfrac{1}{3}x^3,$$
$$\vdots$$

The series solution is given as

$$\begin{aligned} y(x) = {}& 1 + Bx + Bx + \tfrac{4}{3}x^{3/2} + \tfrac{2}{5}Bx^{5/2} + \tfrac{1}{3}x^3 + \tfrac{3}{70}B^2x^{7/2} + \tfrac{2}{15}Bx^4 + \tfrac{2}{27}x^{9/2} \\ & + \tfrac{3}{70}B^2x^{7/2} - \tfrac{1}{252}B^3x^{9/2} + \tfrac{1}{175}B^2x^5 \\ & + \tfrac{2}{27}x^{9/2} + \tfrac{3}{70}B^2x^{7/2} + \tfrac{1}{1056}B^4x^{11/2} + \tfrac{4}{1575}B^3x^6 + \tfrac{557}{100100}B^2x^{13/2} + \cdots. \end{aligned}$$

Setting $x^{1/2} = t$, the series solution is obtained as

$$\begin{aligned} y(t) = {}& 1 + Bt^2 + \tfrac{4}{3}t^3 + \tfrac{2}{5}Bt^5 + \tfrac{1}{3}t^6 + \tfrac{3}{70}B^2t^7 + \tfrac{2}{15}Bt^8 + \left(-\tfrac{1}{252}B^3 + \tfrac{2}{27}\right)t^9 + \\ & \tfrac{1}{175}B^2t^{10} + \left(\tfrac{1}{1056}B^4 + \tfrac{31}{1485}B\right)t^{11} + \left(\tfrac{4}{1575}B^3 + \tfrac{4}{405}\right)t^{12} + \left(-\tfrac{3}{9152}B^5 + \tfrac{557}{100100}B^2\right)t^{13} \\ & + \left(-\tfrac{29}{24255}B^4 + \tfrac{4}{693}B\right)t^{14} + \cdots. \end{aligned}$$

2.2 Example

Consider the following nonlinear boundary value problem of twelfth order

$$y^{(xii)}(x) = \frac{1}{2}e^{-x}y^2(x), 0 < x < 1,$$

Table 1: Pade' approximants and initial slopes $y'(0)$

Pade approximants	Initial slope $y'(0)$	Error (%)
[2/2]	-1.211413729	23.71
[4/4]	-1.550525919	2.36
[7/7]	-1.586021037	12.9 x 10^{-2}
[8/8]	-1.588076820	3.66 x 10^{-4}
[10/10]	-1.588076779	3.64 x 10^{-4}

with boundary conditions

$$y(0) = y''(0) = y^{(iv)}(0) = y^{(vi)}(0) = y^{(viii)}(0) = y^{(x)}(0) = 2,$$
$$y(1) = y''(1) = y^{(iv)}(1) = y^{(vi)}(1) = y^{(viii)}(1) = y^{(x)}(1) = 2e.$$

The exact solution of the problem is

$$y(x) = 2\,e^{x}.$$

The correction functional is given as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s) \left(\frac{d^{12}y_n}{d\,x^{12}} - \frac{1}{2}\,e^{-x}\tilde{y}_n^2(x)\right) ds.$$

Making the correction functional stationary, the Lagrange multiplier is identified as $\lambda(s) = \frac{1}{11!}(s-x)^{11}$, we get the following iterative formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{11!}\,(s-x)^{11} \left(\frac{d^{12}y_n}{d\,x^{12}} - \frac{1}{2}\,e^{-x}y_n^2(x)\right) ds.$$

where

$$A = y'(0), B = y^{(3)}(0), C = y^{(5)}(0), D = y^{(7)}(0), E = y^{(9)}(0), F = y^{(11)}(0).$$

Consequently, following approximants are obtained

$$y_0(x) = 2,$$

$$y_1(x) = 2 + Ax + x^2 + \tfrac{1}{3!}B\,x^3 + \tfrac{2}{4!}x^4 + \tfrac{1}{5!}Cx^5 + \tfrac{2}{6!}x^6$$
$$+\tfrac{1}{7!}Dx^7 + \tfrac{2}{8!}x^8 + \tfrac{1}{9!}Ex^9 + \tfrac{2}{10!}x^{10} + \tfrac{1}{11!}Fx^{11} + \tfrac{2}{12!}x^{12} - \tfrac{2}{13!}x^{13} + \cdots,$$
$$\vdots$$

The series solution is given as

$$y(x) = 2 + Ax + \tfrac{2}{2!}x^2 + \tfrac{1}{3!}Bx^3 + \tfrac{2}{4!}x^4 + \tfrac{1}{5!}Cx^5 + \tfrac{1}{6!}x^6+$$
$$\tfrac{1}{7!}Dx^7 + \tfrac{2}{8!}x^8 + \tfrac{1}{9!}Ex^9 + \tfrac{2}{10!}x^{10} + \tfrac{1}{11!}Fx^{11} + \tfrac{2}{12!}x^{12} + \tfrac{2}{13!}(A-1)\,x^{13} + \cdots.$$

Imposing the boundary conditions at $x = 1$, will yield

$$A = 2.002114383, B = 1.986107510, C = 2.333176702,$$
$$D = 3.001515917, E = 1.986107510, F = 2.102525395.$$

The series solution is given as

$$y(x) = 2 + 2.002114783\,x + x^2 + 0.329322880\,x^3 + \tfrac{1}{12}\,x^4 + 0.01944313918\,x^5$$
$$+\tfrac{1}{720}\,x^6 + 0.0005955388724\,x^7 + \tfrac{1}{20160}\,x^8 + 0.5473179867 \times 10^{-5}\,x^9$$
$$+\tfrac{1}{1814400}\,x^{10} + 0.5267269408 \times 10^{-7}\,x^{11} + \tfrac{1}{239500800}\,x^{12}$$
$$+0.3218601046 \times 10^{-9}\,x^{13} + \cdots.$$

*Error = Exact solution-Series solution.

Table 2: Error estimates

x	Exact solution	Series solution	*Errors
0.0	2.000000000	2.000000000	0.00000000000
0.1	2.2103418362	2.2105496398	0.0002078037
0.2	2.4428055163	2.4431996667	0.0003941504
0.3	2.6997176152	2.7002578996	0.0005402845
0.4	2.9836493953	2.9842815093	0.0006321141
0.5	3.2974425414	3.2981039180	0.0006613766
0.6	3.6442376008	3.6448637056	0.0006261048
0.7	4.0275054149	4.0280359083	0.0005304933
0.8	4.4510818570	4.4514661221	0.0003842651
0.9	4.9192062223	4.9194078617	0.0002016394
1.0	5.4365636569	5.4365636559	0.0002016394

2.3 Example

Consider the following nonlinear boundary value problem of fifth-order,

$$y^{(v)}(x) = e^{-x}y^2(x),$$

with boundary conditions

$$y(0) = y'(0) = y''(0) = 1; \quad y(1) = y'(1) = e.$$

The correction functional with $\lambda\,(s) = \frac{1}{4!}\,(s-x)^4$, is given by

$$y_{n+1}(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}Ax^3 + \frac{1}{4!}Bx^4 + \int_0^x \frac{1}{4!}(s-x)^4\left(\frac{d^5y_n}{dx^5} - e^{-x}y_n^2(x)\right)ds,$$

consequently, following approximants are obtained:

$$y_0(x) = 1,$$

$$y_1(x) = 1 + x^2 + \left(\tfrac{1}{6}A - \tfrac{1}{6}\right)x^3 + \left(\tfrac{1}{24}B + \tfrac{1}{24}\right)x^4 - e^{-x},$$

$$\vdots$$

The series solution is given by:

$$y(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}Ax^3 + \tfrac{1}{24}Bx^4 + \tfrac{1}{120}x^5 + \tfrac{1}{720}x^6 + \tfrac{1}{5040}x^7 + \left(\tfrac{1}{20160}A - \tfrac{1}{40320}\right)x^8$$
$$+ \left(\tfrac{1}{18144} - \tfrac{1}{362880}\right)x^9 + \tfrac{1}{3628800}x^{10} + \left(\tfrac{1}{995840}A^2 - \tfrac{1}{997920}A + \tfrac{1}{1900800}\right)x^{11}$$
$$+ \left(-\tfrac{1}{3421440}A^2 + \tfrac{1}{2280960}A - \tfrac{1}{6842880}B + \tfrac{1}{6842880}AB - \tfrac{101}{479001600}\right)x^{12} + \cdots.$$

Imposing the boundary conditions at $x = 1$ and using $y(1) = y'(1) = e$,we get

A = 0.9999967742, B = 1.0000145020.

Consequently, the series solution is given as

$$y(x) = 1 + x + 0.5x^2 + 0.166666236x^3 + 0.04166727092x^4 + 0.008333333333x^5 +$$
$$0.00138888888x^6 + 0.000198412x^7 + 0.00002480142729x^8 + 0.00005236x^9 +$$
$$0.000000275x^{10} - 0.00000898x^{11} - 0.000000064x^{12} + \cdots,$$

$$^{*}Error = Exact\ solution - Series\ solution.$$

Table 3: Error estimates

x	Exact solution	*Error (VIM)	*Error (B-spline)
0.0	0.000000000000000	0.00000	0.00000
0.1	.099465382626452	-3.0E-11	-8.0E-03
0.2	.195424441303144	-2.2E-10	-1.2E-03
0.3	.283470349583825	-4.0E-10	-5.0E-03
0.4	.358037927419948	-8.0E-10	3.0E-03
0.5	.412180317653458	-1.2E-09	8.0E-03
0.6	.437308512065821	-2E-09	6.0E-03
0.7	.422888068537963	-2.2E-09	-0.0000
0.8	.356086548529305	-1.9E-09	9.0E-03
0.9	.221364279978394	-1.4E-09	-9.0E-03
1.0	.000000000024272	0.00000	0.00000

2.4 Example

Consider the following nonlinear Schrödinger equation

$$iu_t + u_{xx} + 2\,u\,|u|^2 = 0,$$

with initial conditions

$$u(x,0) = e^{ix}.$$

The correction functional with $\lambda(s) = -1$, is given as

$$u_{n+1}(x,\,t) = e^{ix} - \int_0^t \left(i\frac{\partial u_n}{\partial s} + (u_n)_{xx} + 2\,u_n\,|u_n|^2 \right) ds.$$

Consequently, following approximants are obtained

$$u_0(x,t) = e^{ix},$$

$$u_1(x,t) = e^{ix}\,(1 + it)\,,$$

$$\vdots$$

The solution in a series form is given by

$$u(x,t) = e^{3ix}\left(1 + it + \frac{(it)^2}{2!}t^2 + \frac{(it)^3}{3!}t^3 + \frac{(it)^4}{4!}t^4 + \cdots\right),$$

and in a closed form by

$$u(x,t) = e^{i\,(x+t)}.$$

2.5 Example

Consider the following singularly perturbed sixth order Boussinesq equation

$$u_{tt} = u_{xx} + 3(u^2)_{xx} + u_{xxxx},$$

with initial conditions

$$u(x,\,0) = \frac{2a\,k^2 e^{kx}}{(1 + a\,e^{k\,x})^2}, u_t(x,0) = \frac{2\,a\,k^3\sqrt{1+k^2}(1 - a\,e^{kx})\,e^{kx}}{(1 + a\,e^{kx})^3},$$

where a and k are arbitrary constants. The exact solution $u(x,t)$ of the problem is given as

$$u(x,t) = 2\frac{a\,k^2\exp(k\,x + k\sqrt{1+k^2t})}{\left(1 + a\exp(kx + k\sqrt{1+k^2t})\right)^2}.$$

The correction functional with $\lambda = s - x$, is given by

$$u_{n+1}(x,t) = \frac{2\,a\,k^2 e^{kx}}{(1+a\,e^{kx})^2} + \frac{2\,a\,k^3\sqrt{1+k^2}(1-a\,e^{kx})\,e^{kx}}{(1+a\,e^{kx})^3}\,t + \int_0^t (s-x)\left(\frac{\partial^2 u_n}{\partial t^2} - \left((u_n)_{xx} + 3\,(u_n^2)_{xx} + (u_n)_{xxxx}\right)\right) ds.$$

Consequently, following approximants are obtained

$$u_0(x,t) = \frac{2e^x}{(1+e^x)^2},$$

$$u_1(x,t) = \frac{2e^x}{(1+e^x)^2} + \frac{2\,a\,k^3\sqrt{1+k^2}(1-a\,e^{kx})\,e^{kx}}{(1+a\,e^{kx})^3}\,t + \frac{2\,e^x\left(1-4\,e^x+e^{2x}\right)}{(1+e^x)^4}\,t^2,$$

$$\vdots$$

The series solution is given as

$$\begin{aligned}u(x,t) = {} & \frac{2e^x}{(1+e^x)^2} + \frac{2\,a\,k^3\sqrt{1+k^2}(1-a\,e^{kx})\,e^{kx}}{(1+a\,e^{kx})^3}\,t + \frac{2\,e^x\left(1-4\,e^x+e^{2x}\right)}{(1+e^x)^4}\,t^2 \\ & - \frac{2\sqrt{2}\,e^x\,(-1+e^x)\left(1-10\,e^x+e^{2x}\right)}{3(1+e^x)^5}\,t^3 + \frac{e^x\left(1-4e^x+e^{2x}\right)\left(1-44\,e^x+78\,e^{2x}-44\,e^{3x}+e^{4x}\right)}{3\,(1+e^x)^8}\,t^4 \\ & + \frac{8\,e^{2x}\left(1-10\,e^x+20\,e^{2x}-10\,e^{3x}+e^{4x}\right)}{(1+e^x)^8}\,t^4 - \frac{\sqrt{2}\,e^x(-1+e^x)\left(1-56\,e^x+246\,e^{2x}-56\,e^{3x}+e^{4x}\right)}{15\,(1+e^x)^7}\,t^5 \\ & + \frac{e^x\left(1-452\,e^x+19149\,e^{2x}-207936\,e^{3x}+807378\,e^{4x}-1256568\,e^{5x}\right)}{45\,(1+e^x)^{12}}\,t^6 \\ & + \frac{e^x\left(807378\,e^{6x}-207936\,e^{7x}+19149\,e^{8x}-452\,e^{9x}+e^{10x}\right)}{45\,(1+e^x)^{12}}\,t^6 + \cdots,\end{aligned}$$

Table 4: Error estimates

x_i	t_j					
	0.01	0.02	0.04	0.1	0.2	0.5
-1	2.80886 E-14	1.79667 E-12	1.15235 E-10	2.83355 E-8	1.83899 E-6	4.74681 E-4
-0.8	6.27276 E-14	4.01362 E-12	2.57471 E-10	6.33178 E-8	4.10454 E-6	1.04489 E-3
-0.6	6.08402 E-14	3.90188 E-12	2.25663 E-10	6.18024 E-8	4.02299 E-6	1.03093 E-3
-0.4	1.16573 E-14	7.41129 E-13	4.82756 E-11	1.23843 E-8	8.53800 E-6	2.46302 E-4
-0.2	5.53446 E-14	3.53395 E-12	2.25663 E-10	5.47485 E-8	3.47264 E-6	8.35783 E-4
0	8.63198 E-14	5.53357 E-12	2.54174 E-10	8.65197 E-8	5.54893 E-6	1.37353 E-3
0.2	5.56222 E-14	3.55044 E-12	2.27779 E-10	5.60362 E-8	3.63600 E-6	9.29612 E-4
0.4	1.14353 E-14	7.14928 E-13	4.49107 E-11	1.03370 E-8	5.93842 E-7	9.61260 E-5
0.6	6.06182 E-14	3.87551 E-12	2.47218 E-10	5.97562 E-8	3.76275 E-6	8.79002 E-4
0.8	6.23945 E-14	3.99519 E-12	2.55127 E-10	6.18881 E-8	3.92220 E-6	9.36404 E-4
1	2.79776 E-14	1.78946 E-12	1.14307 E-10	2.77684 E-8	1.76607 E-6	4.28986 E-4

2.6 Example

Consider the following telegraph equation

$$u_{xx} = u_{tt} + u_t - u,$$

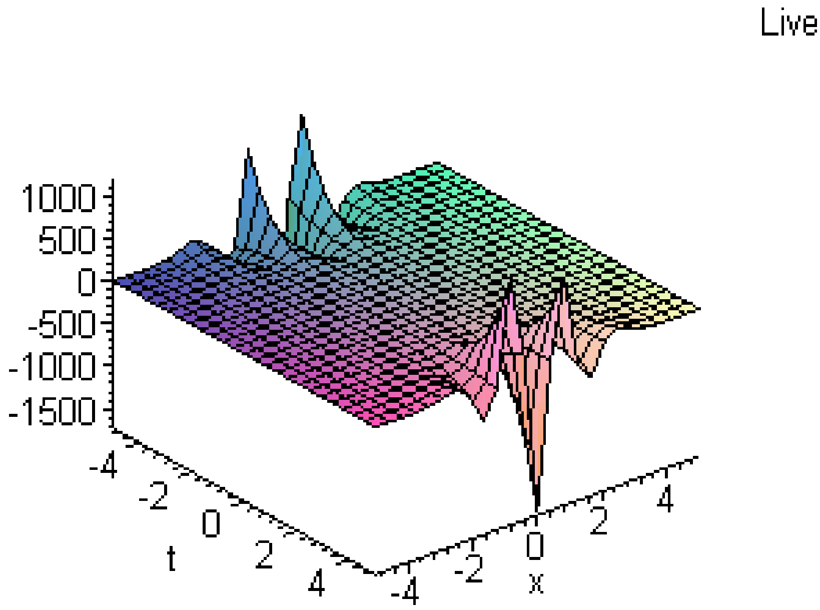

Figure 1: The truncated VIM series solution for example 2.5.

with boundary conditions

$$u(0,\, t)\; =\; e^{-2t}, u_x(0,\, t)\; =\; e^{-2t},$$

and the initial conditions

$$u\,(x,\,0)\;=e^{x}, u_t\,(x,\,0)\;=\;-2\,e^{x}.$$

The correction functional with $\lambda\,(s)\;=\;(s-\;t)$,is given by

$$u_{n+1}\,(x,\,t)\;=\;e^{-2t}+xe^{-2t}\;+\int\limits_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2}-\frac{\partial^2 u_n}{\partial t^2}-\frac{\partial\, u_n}{\partial t}+u_n\right)\,ds.$$

Consequently, following approximants are obtained

$$u_0\,(x,\,t)\;=\;e^{-2t}\,(1+x)\,,$$

$$u_1\,(x,\,t)\;=\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3\right)\;e^{-2t},$$

$$\vdots$$

The series solution is given by

$$u\,(x,\,t)\;=\;\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7+\frac{1}{8!}x^8+\frac{1}{9!}x^9+\cdots\right)\;e^{-2t},$$

and the closed form solution is given as

$$u\,(x,\,t)\;=\;e^{x\,-2t},$$

which is the exact solution.

2.7 Example

Consider the following linear inhomogeneous Klein-Gordon equation

$$u_{tt}(x,\,t)\,-\,u_{xx}(x,\,t)\,+u(x,\,t)\,=\,2\,\sin x,$$

with initial conditions

$$u(x,\,0)\,=\sin x, u_t(x,\,0)\,=1.$$

The correction functional with $\lambda(s)\,=\,(s-\,t)$, is given by

$$u_{n+1}(x,\,t)\,=\,\sin x\,+t\,+\int_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2}\,-\,\frac{\partial^2 u_n}{\partial x^2}+\,u_n(s,\,t)-\,2\sin s\right)ds.$$

Consequently, following approximants are obtained

$$u_0(x,\,t)\,=\,\sin x\,+t\,+t^2\sin x,$$

$$u_1(x,\,t)\,=\,\sin x\,+t\,+t^2\sin x-t^2\sin x-\,\tfrac{1}{3!}t^4\sin x-\,\tfrac{1}{3!}t^3,$$

$$\vdots$$

The series solution is given by

$$u(x,\,t)\,=\,\sin x\,+t\,+t^2\sin x-t^2\sin x-\,\frac{1}{3!}t^4\sin x-\,\frac{1}{3!}t^3+\,\frac{1}{3!}t^4\sin x+\,\frac{1}{90}t^6\sin x+\,\frac{1}{5!}t^5+\cdots,$$

and the closed form solution is given as

$$u(x,\,t)\,=\,\sin x\,\,+\sin t.$$

2.8 Example

Consider the following Sine-Gordon equation

$$u_{tt}(x,\,t)\,-\,u_{xx}(x,\,t)\,\,=\,\sin\,u,$$

with initial conditions

$$u(x,\,0)\,\,=\frac{\pi}{2}, u_t(x,\,0)\,=1.$$

The correction functional with $\lambda(s)\,=\,(s-\,t)$, is given by

$$u_{n+1}(x,\,t)\,=\,\frac{\pi}{2}+t+\int_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2}\,-\,\frac{\partial^2 u_n}{\partial x^2}-\,\sin\,u_n\right)ds.$$

Consequently, following approximants are obtained

$$u_0(x,\,t)\,=\,\frac{\pi}{2}+t,$$

$$u_1(x,\,t)\,=\,\tfrac{\pi}{2}+t+1-\cos\,t,$$

$$\vdots$$

The series solution is given by

$$u(x,\,t)\,=\,\frac{\pi}{2}+t+\frac{1}{2!}t^2-\frac{1}{4!}t^4+\cdots.$$

3 Conclusion

In this paper, we applied VIM for solving a variety of various initial and boundary value problems. The proposed VIM is employed without using linearization, perturbation discretization or restrictive assumptions. Moreover, the proposed technique is easier to implement and is more user friendly as compare to decomposition and other methods which involve lot of complexities coupled with the huge computational work.

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