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Controllability of Fractional Stochastic Delay Equations

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Abstract: Sufficient conditions for controllability of fractional stochastic delay equations are established. The results are obtained by using a stochastic version of the well known Banach fixed point theorem and semigroup theory.

Keywords:controllability; fractional integrals; stochastic delay equations; fixed point theorem
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1 Introduction

Stochastic delay equations serve as an abstract formulation of many partial differential equations that arise in problems of heat flow in materials with memory, viscoelasticity, and many other physical phenomena (see [1], [2]). The main objective of this paper is to derive controllability conditions for semilinear fractional stochastic delay equations in Hilbert space of the form

$$\begin{aligned} x(t) = \psi(0) - \frac{1}{\Gamma(\alpha)}\int_0^t \frac{Ax(s)}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(Bu)(s)}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s,x(s),x(s-\tau(s)))}{(t-s)^{1-\alpha}}ds \\ + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(s,x(s),x(s-\tau(s)))}{(t-s)^{1-\alpha}}d\omega(s), \ \ t\in J=[0,T], \\ x(t)=\psi(t), \ \ t\in[-r,0], \end{aligned} \tag{1.1}$$

where $0<\alpha\le 1$, $T>0$ and A is a linear closed operator , defined on a given Hilbert space X. It is assumed that A generates an analytic semigroup $S(t)$, $t\ge 0$. The state $x(.)$ takes its values in the Hilbert space X, and the control function $u(.)$ is in $L^2(J,U)$, the Hilbert space of admissible control functions with Ua Hilbert space. B is a bounded linear operator from U into X.
Let K be a separable Hilbert space, and let (Ω,F,F_t,P) be a complete probability space furnished with a complete family of right continuous increasing sigma algebras$\{F_t\}$ satisfying $F_t\subset F$ for $t\ge 0$. The process $\{\omega(t),t\ge 0\}$ is a K-valued, F_t-adapted Brownian motion with $P\{\omega(0)=0\}=1$, and $\psi(.)$ is an X-valued F_0-measurable random variable independent of the Brownian motion $\omega(.)$. For any Banach space Y, let $L_2(\Omega,Y)$ denote the space of strongly measurable, Y-valued, square integrable random variables equipped with the norm topology $\parallel x\parallel_{L_2(\Omega,Y)}=\{E\parallel x\parallel_Y^2\}^{\frac{1}{2}}$, where E is defined as integration with respect to the probability measure P. Then $L_2(\Omega,Y)$ is also Hilbert space since Y is a Hilbert space. Let $\tau(.)$ be a continuous nonnegative function on R^+ and define $r=\sup\{\tau(t)-t: t\ge 0\}<\infty$. Let $\psi\in L_2^0([-r,0],X_\gamma)$, the family of all continuous square integrable stochastic processes $\psi(.)$ such that $\sup\{E\parallel\psi\parallel_\gamma^2\}<\infty$, for $-r\le t\le 0$. Let $I=[-r,T]$ and $M(I,Y)$ denote the space of F_t-adapted stochastic processes defined on I, taking values in Y, having square integrable norms, that are continuous in t on I in the mean square sense. This is a Banach space with respect to the norm topology

$$\parallel\zeta\parallel_{M(I,Y)}=\{\sup_{t\in I}E\parallel\zeta(t)\parallel_Y^2\}^{\frac{1}{2}}, \ \ \zeta\in M(I,Y). \tag{1.2}$$

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2 Controllability results

In this section we shall use the fixed point technique to study the controllability of semilinear fractional stochastic delay equations (see [3], [4], [5]). Let us assume the following conditions :
(i) for $0 \leq \gamma < 1$, $X_\gamma = [D(A^\gamma)]$ is a Banach space with respect to the graph topology induced by the graph norm

$$\| x \|_\gamma = \| A^\gamma x \| + \| x \|, \; for \; x \in D(A^\gamma); \tag{2.1}$$

(ii) the function f maps X_γ to X and there exists a constant $C > 0$ such that

$$\begin{aligned} &\| f(t,x,y) - f(t,\tilde{x},\tilde{y}) \|_X \leq C(\| x - \tilde{x} \|_\gamma + \| y - \tilde{y} \|_\gamma), \\ &\| f(t,x,y) \|_X \leq C\{1 + \| x \|_\gamma + \| y \|_\gamma\} \; \forall \, x, \; y, \; \tilde{x}, \; \tilde{y} \in X_\gamma; \end{aligned} \tag{2.2}$$

(iii) the function g maps X_γ to $L(K,X)$ and there exists a constant $C > 0$ such that

$$\begin{aligned} &\| g(t,x,y) - g(t,\tilde{x},\tilde{y}) \|_{L(K,X)} \leq C(\| x - \tilde{x} \|_\gamma + \| y - \tilde{y} \|_\gamma), \\ &\| g(t,x,y) \|_{L(K,X)} \leq C\{1 + \| x \|_\gamma + \| y \|_\gamma\} \; \forall \, x, \; y, \; \tilde{x}, \; \tilde{y} \in X_\gamma; \end{aligned} \tag{2.3}$$

(iv) the linear operator W from $L^2(J,U)$ into X defined by

$$Wu = \alpha \int_0^T \int_0^\infty \theta (T-s)^{\alpha-1} \xi_\alpha(\theta) S((T-s)^\alpha \theta) Bu(s) d\theta ds \qquad (2.4),$$

$\xi_\alpha(\theta)$ is a probability density function defined on $[0,\infty)$ (see [6], [7]) , has an invertible operator W^{-1} defined on $X \backslash kerW$ and there exist the positive constants $N_1, \; N_2$ such that

$$\| B \|^2 \leq N_1, \quad \| W^{-1} \|^2 \leq N_2. \tag{2.4}$$

Here, $L(K,X)$ is the family of all bounded linear operators from K into X, equipped with the usual operator norm topology, and ω is a F_t-adapted Brownian motion having a nuclear covariance operator $Q \in L_n^+(Y)$. By the assumptions (i), (ii) and (iii), there exists a unique stochastic process $x(.) \in M(I, X_\gamma)$, that is, a solution of (1.1) (see [8], [9], [10]) such that x(.) is F_t-adapted, measurable, and almost surely that $\int_{-r}^T \| x(s) \|_\gamma^2 < \infty$, with

$$x(t) = \int_0^\infty \xi_\alpha(\theta) S(t^\alpha \theta) \psi(0) d\theta + \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) S((t-s)^\alpha \theta)[(Bu)(s) + f(s,x(s),x(s-\tau(s)))] d\theta ds$$

$$+ \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) S((t-s)^\alpha \theta) g(s,x(s),x(s-\tau(s))) d\theta d\omega(s), \;\; t \geq 0, \tag{2.6}$$

$$x(t) = \psi(t), \;\; t \in [-r, 0].$$

Definition 2.1 *The stochastic system (1.1) is said to be controllable on J, if for every continuous initial random process $\psi(.)$ defined on $[-r,0]$, there exists a control $u \in L^2(J,U)$ such that the solution of (1.1) satisfies $x(T) = x_1$, where x_1 and T are preassigned terminal state and time, respectively. If the system is controllable for all x_1 at $t = T$, it is called completely controllable on J.*

Theorem 2.1 *Suppose that conditions (i), (ii), (iii) and (iv) are satisfied, then system (1.1) is completely controllable on J.*

Proof. Using assumption (iv), define the control

$$u(t) = W^{-1}\{x_1 - \int_0^\infty \xi_\alpha(\theta) S(T^\alpha \theta) \psi(0) d\theta - \alpha \int_0^T \int_0^\infty \theta (T-s)^{\alpha-1} \xi_\alpha(\theta) S((T-s)^\alpha \theta) f(s,x(s),x(s-\tau(s))) d\theta ds$$

$$- \alpha \int_0^T \int_0^\infty \theta (T-s)^{\alpha-1} \xi_\alpha(\theta) S((T-s)^\alpha \theta) g(s,x(s),x(s-\tau(s))) d\theta d\omega(s)\}(t). \tag{2.7}$$

Now, it is shown that when using this control the operator defined by

$$(\Phi x)(t)=\int_0^{\infty}\xi_\alpha(\theta)S(t^\alpha\theta)\psi(0)d\theta+\alpha\int_0^t\int_0^{\infty}\mu(t-\eta)^{\alpha-1}\xi_\alpha(\mu)S((t-\eta)^\alpha\mu)BW^{-1}\{x_1-\int_0^{\infty}\xi_\alpha(\theta)S(T^\alpha\theta)\psi(0)d\theta$$
$$-\alpha\int_0^T\int_0^{\infty}\theta(T-s)^{\alpha-1}\xi_\alpha(\theta)S((T-s)^\alpha\theta)f(s,x(s),x(s-\tau(s)))d\theta ds$$
$$-\alpha\int_0^T\int_0^{\infty}\theta(T-s)^{\alpha-1}\xi_\alpha(\theta)S((T-s)^\alpha\theta)g(s,x(s),x(s-\tau(s)))d\theta d\omega(s)\}(\eta)d\mu d\eta$$
$$+\alpha\int_0^t\int_0^{\infty}\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)f(s,x(s),x(s-\tau(s)))d\theta ds$$
$$+\alpha\int_0^t\int_0^{\infty}\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)g(s,x(s),x(s-\tau(s)))d\theta d\omega(s),\quad t\in J, \tag{2.8}$$

$(\Phi x)(t)=\psi(t),\quad -r\le t\le 0,$
has fixed point. This fixed point is a solution of (1.1). Clearly $(\Phi x)(0)=\psi(0)$, which means that the control $u(.)$ steers the semilinear fractional stochastic delay equations from the initial state $\psi(.)$ to x_1 in time T provided the nonlinear operator Φ has a fixed point. First, it must be shown that Φ maps $M(I,X_\gamma)$ into $M(I,X_\gamma)$. Without loss of generality, assume that $0\in\rho(A)$. Otherwise, if $0\notin\rho(A)$, for the identity operator I add the term λI to A giving $A_\lambda=A+\lambda I$, then $0\in\rho(A_\lambda)$. This simplifies the graph norm to $\|\chi\|_\gamma=\|A^\gamma\chi\|$ for $\chi\in D(A^\gamma)$. Since $S(t)$, $t\ge 0$ is an analytic semigroup and A^γ is a closed operator, there exist numbers $C_1\ge 1$ and C_γ such that

$$\sup_{t\in J}\|S(t)\|^2_{L(X)}\le C_1,\quad \|A^\gamma S(t)\|_{L(X)}\le C_\gamma t^{-\gamma},\ for\ \ t\ge 0. \tag{2.9}$$

Further, $|a+b+c|^2\le 9(|a|^2+|b|^2+|c|^2)$ for any real numbers a,b,c. Hence, for $x\in(I,X_\gamma)$,

$$E(\sup_{t\in[-r,0]}\|\psi(t)\|^2_X)<\infty,\ for\ \ -r\le t\le 0, \tag{2.10}$$

and for $t\in J$

$$E(\sup_{t\in J}\|(\Phi x)(t)\|^2_X)\le 9\sup_{t\in J}E(\|\int_0^{\infty}\xi_\alpha(\theta)S(t^\alpha\theta)\psi(0)d\theta\|^2_\gamma)$$
$$+9E\|\alpha\int_0^t\int_0^{\infty}\mu(t-\eta)^{\alpha-1}\xi_\alpha(\mu)S((t-\eta)^\alpha\mu)BW^{-1}\{x_1-\int_0^{\infty}\xi_\alpha(\theta)S(T^\alpha\theta)\psi(0)d\theta$$
$$-\alpha\int_0^T\int_0^{\infty}\theta(T-s)^{\alpha-1}\xi_\alpha(\theta)S((T-s)^\alpha\theta)f(s,x(s),x(s-\tau(s)))d\theta ds$$
$$-\alpha\int_0^T\int_0^{\infty}\theta(T-s)^{\alpha-1}\xi_\alpha(\theta)S((T-s)^\alpha\theta)g(s,x(s),x(s-\tau(s)))d\theta d\omega(s)\}(\eta)d\mu d\eta\|^2_\gamma$$
$$+9E\|\alpha\int_0^t\int_0^{\infty}\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)f(s,x(s),x(s-\tau(s)))d\theta ds\|^2_\gamma$$
$$+9E\|\alpha\int_0^t\int_0^{\infty}\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)S((t-s)^\alpha\theta)g(s,x(s),x(s-\tau(s)))d\theta d\omega(s)\|^2_\gamma$$
$$\le 9\sup_{t\in J}E(\|A^\gamma S(t^\alpha\theta)\psi(0)\|^2_X)$$
$$+9N_1N_2\alpha^2\int_0^t(t-\eta)^{2(\alpha-1)}\|A^\gamma S((t-\eta)^\alpha\mu)\|^2_{L(X)}\{E\|x_1\|^2_\gamma+E\|A^\gamma S(T^\alpha\theta)\psi(0)\|^2_X$$
$$+\alpha^2(\int_0^T(T-s)^{2(\alpha-1)}\|A^\gamma S((T-s)^\alpha\theta)\|^2_{L(X)}ds)E\int_0^T\|f(s,x(s),x(s-\tau(s)))\|^2_X ds$$
$$+\alpha^2Tr(Q)\int_0^T(T-s)^{2(\alpha-1)}E\|A^\gamma S((T-s)^\alpha\theta)g(s,x(s),x(s-\tau(s)))\|^2_{L(K,X)}ds\}d\eta$$
$$+9\alpha^2(\int_0^t(t-s)^{2(\alpha-1)}\|A^\gamma S((t-s)^\alpha\theta)\|^2_{L(X)})E\int_0^t\|f(s,x(s),x(s-\tau(s)))\|^2_X ds$$

$$+ 9Tr(Q)\alpha^2 E\int_0^t (t-s)^{2(\alpha-1)} \parallel A^\gamma S((t-s)^\alpha\theta)g(s,x(s),x(s-\tau(s))) \parallel^2_{L(K,X)} ds$$

from (ii), (iii) and (2.9) yields

$$E(\sup_{t\in J} \parallel (\Phi x)(t) \parallel^2_X) \leq 9C_1E(\parallel \psi(0) \parallel^2_\gamma) + 9N_1N_2\alpha^2C^2_\gamma \int_0^T (T-\eta)^{2\alpha(1-\gamma)-2}d\eta\{E \parallel x_1 \parallel^2_\gamma + C_1E \parallel \psi(0) \parallel^2_\gamma$$

$$+ \alpha^2C^2_\gamma C^2(\int_0^T (T-s)^{2\alpha(1-\gamma)-2}ds)T(1+ \sup_{0\leq s\leq t} E \parallel x(s) \parallel^2_\gamma + \sup_{0\leq s\leq t} E \parallel x(s-\tau(s)) \parallel^2_\gamma)$$

$$+\alpha^2Tr(Q)(2C^2_\gamma C^2)\int_0^T (T-s)^{2\alpha(1-\gamma)-2}(1+ \sup_{0\leq s\leq t} E \parallel x(s) \parallel^2_\gamma + \sup_{0\leq s\leq t} E \parallel x(s-\tau(s)) \parallel^2_\gamma)ds\}$$

$$+ 9\alpha^2C^2_\gamma C^2(\int_0^T (T-s)^{2\alpha(1-\gamma)-2}ds)T(1+ \sup_{0\leq s\leq t} E \parallel x(s) \parallel^2_\gamma + \sup_{0\leq s\leq t} E \parallel x(s-\tau(s)) \parallel^2_\gamma)$$

$$+9\alpha^2Tr(Q)(2C^2_\gamma C^2)\int_0^T (T-s)^{2\alpha(1-\gamma)-2}(1+ \sup_{0\leq s\leq t} E \parallel x(s) \parallel^2_\gamma + \sup_{0\leq s\leq t} E \parallel x(s-\tau(s)) \parallel^2_\gamma)ds.$$

Let $\alpha(1-\gamma) = \beta$ where $\frac{1}{2} < \beta \leq 1$, then

$$E(\sup_{t\in J} \parallel (\Phi x)(t) \parallel^2_X) \leq 9C_1E(\parallel \psi(0) \parallel^2_\gamma) + 9N_1N_2\alpha^2C^2_\gamma \frac{T^{2\beta-1}}{2\beta-1}\{E \parallel x_1 \parallel^2_\gamma + C_1E \parallel \psi(0) \parallel^2_\gamma$$

$$+ \alpha^2C^2_\gamma C^2\frac{T^{2\beta}}{2\beta-1}(1+ \sup_{0\leq s\leq t} E \parallel x(s) \parallel^2_\gamma + \sup_{0\leq s\leq t} E \parallel x(s-\tau(s)) \parallel^2_\gamma)$$

$$+ \alpha^2Tr(Q)(2C^2_\gamma C^2)\frac{T^{2\beta-1}}{2\beta-1}(1+ \sup_{0\leq s\leq t} E \parallel x(s) \parallel^2_\gamma + \sup_{0\leq s\leq t} E \parallel x(s-\tau(s)) \parallel^2_\gamma)\}$$

$$+ 9\alpha^2C^2_\gamma C^2\frac{T^{2\beta}}{2\beta-1}(1+ \sup_{0\leq s\leq t} E \parallel x(s) \parallel^2_\gamma + \sup_{0\leq s\leq t} E \parallel x(s-\tau(s)) \parallel^2_\gamma)$$

$$+ 9\alpha^2Tr(Q)(2C^2_\gamma C^2)\frac{T^{2\beta-1}}{2\beta-1}(1+ \sup_{0\leq s\leq t} E \parallel x(s) \parallel^2_\gamma + \sup_{0\leq s\leq t} E \parallel x(s-\tau(s)) \parallel^2_\gamma)$$

$$\leq 9C_1E(\parallel \psi(0) \parallel^2_\gamma) + 9N_1N_2\alpha^2C^2_\gamma \frac{T^{2\beta-1}}{2\beta-1}\{E \parallel x_1 \parallel^2_\gamma + C_1E \parallel \psi(0) \parallel^2_\gamma$$

$$+ \alpha^2C^2_\gamma C^2\frac{T^{2\beta}}{2\beta-1}(1+2 \parallel x \parallel^2_{M(I,X_\gamma)}) + \alpha^2Tr(Q)(2C^2_\gamma C^2)\frac{T^{2\beta-1}}{2\beta-1}(1+2 \parallel x \parallel^2_{M(I,X_\gamma)})\}$$

$$+ 9\alpha^2C^2_\gamma C^2\frac{T^{2\beta}}{2\beta-1}(1+2 \parallel x \parallel^2_{M(I,X_\gamma)}) + 9\alpha^2Tr(Q)(2C^2_\gamma C^2)\frac{T^{2\beta-1}}{2\beta-1}(1+2 \parallel x \parallel^2_{M(I,X_\gamma)})$$

$$\leq 9C_1E(\parallel \psi(0) \parallel^2_\gamma) + 9N_1N_2\alpha^2C^2_\gamma \frac{T^{2\beta-1}}{2\beta-1}\{E \parallel x_1 \parallel^2_\gamma + C_1E \parallel \psi(0) \parallel^2_\gamma + \Delta(Tr(Q))\} + 9\Delta(Tr(Q)), \quad (2.11)$$

where $Tr(Q)$ represents the trace of the operator Q and

$$\Delta(Tr(Q)) = (\alpha C_\gamma C)^2(T+2Tr(Q))\frac{T^{2\beta-1}}{2\beta-1}(1+2 \parallel x \parallel^2_{M(I,X_\gamma)}) \quad (2.12)$$

Hence

$$\sup_{t\in I} \parallel (\Phi x)(t) \parallel^2_\gamma < \infty \ \ for \ \ x \in M(I,X_\gamma).$$

Since $\psi(.)$ is continuous in $[-r,0]$, to complete the proof it remains to show that $\Phi \in C((0,T),L_2(\Omega,X_\gamma))$. To accomplish that, let $t\in(0,T),\ h>0$ and $t+h\in J$, we have

$$E\{\parallel (\Phi x)(t+h)-(\Phi x)(t) \parallel^2_\gamma\} \leq 9E(\parallel \int_0^\infty \xi_\alpha(\theta)(A^\gamma S((t+h)^\alpha\theta)-A^\gamma S(t^\alpha\theta))\psi(0)d\theta \parallel^2_\gamma)$$

$$+9\alpha^2E \parallel \int_0^t\int_0^\infty \mu\xi_\alpha(\mu)((t+h-\eta)^{\alpha-1}A^\gamma S((t+h-\eta)^\alpha\mu)-(t-\eta)^{\alpha-1}A^\gamma S((t-\eta)^\alpha\mu))BW^{-1}\{x_1$$

$$-\int_0^\infty \xi_\alpha(\theta)S(T^\alpha\theta)\psi(0)d\theta-\alpha\int_0^T\int_0^\infty \theta(T-s)^{\alpha-1}\xi_\alpha(\theta)S((T-s)^\alpha\theta)f(s,x(s),x(s-\tau(s)))d\theta ds$$

$$-\alpha\int_0^T\int_0^\infty \theta(T-s)^{\alpha-1}\xi_\alpha(\theta)S((T-s)^\alpha\theta)g(s,x(s),x(s-\tau(s)))d\theta d\omega(s)\}(\eta)d\mu d\eta\parallel_\gamma^2$$

$$+9\alpha^2E\parallel\int_t^{t+h}\int_0^\infty \mu\xi_\alpha(\mu)(t+h-\eta)^{\alpha-1}A^\gamma S((t+h-\eta)^\alpha\mu)BW^{-1}\{x_1-\int_0^\infty \xi_\alpha(\theta)S(T^\alpha\theta)\psi(0)d\theta$$

$$-\alpha\int_0^T\int_0^\infty \theta(T-s)^{\alpha-1}\xi_\alpha(\theta)S((T-s)^\alpha\theta)f(s,x(s),x(s-\tau(s)))d\theta ds$$

$$-\alpha\int_0^T\int_0^\infty \theta(T-s)^{\alpha-1}\xi_\alpha(\theta)S((T-s)^\alpha\theta)g(s,x(s),x(s-\tau(s)))d\theta d\omega(s)\}(\eta)d\mu d\eta\parallel_\gamma^2$$

$$+9\alpha^2E\parallel\int_0^t\int_0^\infty \theta\xi_\alpha(\theta)((t+h-s)^{\alpha-1}A^\gamma S((t+h-s)^\alpha\theta)-(t-s)^{\alpha-1}A^\gamma S((t-s)^\alpha\theta))f(s,x(s),x(s-\tau(s)))d\theta ds\parallel_\gamma^2$$

$$+9\alpha^2E\parallel\int_t^{t+h}\int_0^\infty \theta\xi_\alpha(\theta)(t+h-s)^{\alpha-1}A^\gamma S((t+h-s)^\alpha\theta)f(s,x(s),x(s-\tau(s)))d\theta ds\parallel_\gamma^2+9\alpha^2E\parallel\int_0^t\int_0^\infty \theta\xi_\alpha(\theta)$$

$$((t+h-s)^{\alpha-1}A^\gamma S((t+h-s)^\alpha\theta)-(t-s)^{\alpha-1}A^\gamma S((t-s)^\alpha\theta))g(s,x(s),x(s-\tau(s)))d\theta d\omega(s)\parallel_{L(K,X)}^2$$

$$+9\alpha^2E\parallel\int_t^{t+h}\int_0^\infty \theta\xi_\alpha(\theta)(t+h-s)^{\alpha-1}A^\gamma S((t+h-s)^\alpha\theta)g(s,x(s),x(s-\tau(s)))d\theta d\omega(s)\parallel_{L(K,X)}^2$$

$$\le 9E(\parallel(A^\gamma S((t+h)^\alpha\theta)-A^\gamma S(t^\alpha\theta))\psi(0)\parallel_\gamma^2)$$

$$+9\alpha^2N_1N_2\int_0^t\parallel(t+h-\eta)^{\alpha-1}A^\gamma S((t+h-\eta)^\alpha\mu)-(t-\eta)^{\alpha-1}A^\gamma S((t-\eta)^\alpha\mu)\parallel_\gamma^2\{E\parallel x_1\parallel_\gamma^2+C_1E\parallel\psi(0)\parallel_\gamma^2$$

$$+(\alpha C_\gamma C)^2\frac{T^{2\beta}}{2\beta-1}(1+\sup_{0\le s\le t}E\parallel x(s)\parallel_\gamma^2+\sup_{0\le s\le t}E\parallel x(s-\tau(s))\parallel_\gamma^2)+2Tr(Q)(\alpha C_\gamma C)^2\frac{T^{2\beta-1}}{2\beta-1}(1+\sup_{0\le s\le t}E\parallel x(s)\parallel_\gamma^2$$

$$+\sup_{0\le s\le t}E\parallel x(s-\tau(s))\parallel_\gamma^2)\}d\eta+9\alpha^2N_1N_2C_\gamma^2\frac{h^{2\beta-1}}{2\beta-1}\{E\parallel x_1\parallel_\gamma^2+C_1E\parallel\psi(0)\parallel_\gamma^2$$

$$+\alpha^2C_\gamma^2C^2\frac{T^{2\beta}}{2\beta-1}(1+\sup_{0\le s\le t}E\parallel x(s)\parallel_\gamma^2+\sup_{0\le s\le t}E\parallel x(s-\tau(s))\parallel_\gamma^2)$$

$$+\alpha^2Tr(Q)(2C_\gamma^2C^2)\frac{T^{2\beta-1}}{2\beta-1}(1+\sup_{0\le s\le t}E\parallel x(s)\parallel_\gamma^2+\sup_{0\le s\le t}E\parallel x(s-\tau(s))\parallel_\gamma^2)\}$$

$$+9\alpha^2C^2\int_0^t\parallel(t+h-s)^{\alpha-1}A^\gamma S((t+h-s)^\alpha\theta)-(t-s)^{\alpha-1}A^\gamma S((t-s)^\alpha\theta)\parallel_\gamma(1+\sup_{0\le s\le t}E\parallel x(s)\parallel_\gamma^2$$

$$+\sup_{0\le s\le t}E\parallel x(s-\tau(s))\parallel_\gamma^2)ds+9\alpha^2C_\gamma^2C^2\frac{h^{2\beta}}{2\beta-1}(1+\sup_{0\le s\le t}E\parallel x(s)\parallel_\gamma^2+\sup_{0\le s\le t}E\parallel x(s-\tau(s))\parallel_\gamma^2)$$

$$+18Tr(Q)\alpha^2C^2\int_0^t\parallel(t+h-s)^{\alpha-1}A^\gamma S((t+h-s)^\alpha\theta)-(t-s)^{\alpha-1}A^\gamma S((t-s)^\alpha\theta)\parallel_\gamma^2ds(1+\sup_{0\le s\le t}E\parallel x(s)\parallel_\gamma^2$$

$$+\sup_{0\le s\le t}E\parallel x(s-\tau(s))\parallel_\gamma^2)+9\alpha^2Tr(Q)(2C_\gamma^2C^2)\frac{h^{2\beta-1}}{2\beta-1}(1+\sup_{0\le s\le t}E\parallel x(s)\parallel_\gamma^2+\sup_{0\le s\le t}E\parallel x(s-\tau(s))\parallel_\gamma^2)$$

$$\le 9E(\parallel(A^\gamma S((t+h)^\alpha\theta)-A^\gamma S(t^\alpha\theta))\psi(0)\parallel_\gamma^2)$$

$$+9\alpha^2N_1N_2\int_0^t\parallel(t+h-\eta)^{\alpha-1}A^\gamma S((t+h-\eta)^\alpha\mu)-(t-\eta)^{\alpha-1}A^\gamma S((t-\eta)^\alpha\mu)\parallel_\gamma^2\{E\parallel x_1\parallel_\gamma^2+C_1E\parallel\psi(0)\parallel_\gamma^2$$

$$+(\alpha C_\gamma C)^2\frac{T^{2\beta}}{2\beta-1}(1+2\parallel x\parallel_{M(I,X\gamma)}^2)+2Tr(Q)(\alpha C_\gamma C)^2\frac{T^{2\beta-1}}{2\beta-1}(1+2\parallel x\parallel_{M(I,X\gamma)}^2)\}d\eta$$

$$+9\alpha^2N_1N_2C_\gamma^2\frac{h^{2\beta-1}}{2\beta-1}\{E\parallel x_1\parallel_\gamma^2+C_1E\parallel\psi(0)\parallel_\gamma^2+\alpha^2C_\gamma^2C^2\frac{T^{2\beta}}{2\beta-1}(1+2\parallel x\parallel_{M(I,X\gamma)}^2)$$

$$+\alpha^2Tr(Q)(2C_\gamma^2C^2)\frac{T^{2\beta-1}}{2\beta-1}(1+2\parallel x\parallel_{M(I,X\gamma)}^2)\}$$

$$+9\alpha^2C^2\int_0^t \| (t+h-s)^{\alpha-1}A^\gamma S((t+h-s)^\alpha\theta)-(t-s)^{\alpha-1}A^\gamma S((t-s)^\alpha\theta) \|_\gamma (1+2\| x \|^2_{M(I,X_\gamma)})ds$$

$$+9\alpha^2C_\gamma^2C^2\frac{h^{2\beta}}{2\beta-1}(1+2\| x \|^2_{M(I,X_\gamma)})$$

$$+18Tr(Q)\alpha^2C^2\int_0^t \| (t+h-s)^{\alpha-1}A^\gamma S((t+h-s)^\alpha\theta)-(t-s)^{\alpha-1}A^\gamma S((t-s)^\alpha\theta) \|^2_\gamma (1+2\| x \|^2_{M(I,X_\gamma)})ds$$

$$+9\alpha^2Tr(Q)(2C_\gamma^2C^2)\frac{h^{2\beta-1}}{2\beta-1}(1+2\| x \|^2_{M(I,X_\gamma)}), \quad (2.13)$$

for $t\in(0,T)$. Thus, letting $h\to 0$, the desired continuity follows. Hence Φ maps $M(I,X_\gamma)$ into itself.
Now, it is shown that for sufficiently small T, defining the interval I leads to a contraction in $M(I,X_\gamma)$. Indeed, for $x,\ y\ \in M(I,X_\gamma)$ satisfying $x(t)=y(t)=\psi(t)$ for $-r\le t\le 0$ it can be easily seen that

$$\sup_{t\in J} E \| (\Phi x)(t)-(\Phi y)(t) \|^2 \le \tilde{K}\sup_{t\in J} E \| x(t)-y(t) \|^2_\gamma$$

where

$$\tilde{K}=9N_1N_2\alpha^2C_\gamma^2[\alpha C_\gamma C]^2(T+2Tr(Q))\frac{T^{4\beta-2}}{(2\beta-1)^2}+9[\alpha C_\gamma C]^2(T+2Tr(Q))\frac{T^{2\beta-1}}{(2\beta-1)}. \quad (2.14)$$

Thus, for sufficiently small T, $\tilde{K}<1$ and Φ is a contraction in $M(I,X_\gamma)$ and so, by the Banach fixed point theorem, Φ has a unique fixed point $x\in M(I,X_\gamma)$. Any fixed point of Φ is a solution of (1.1) on J satisfying $(\Phi x)(t)=x(t)\in X$, for all $\psi(.)$ and $T>0$. Thus, system (1.1) is completely controllable on J.

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