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On the Generalized Nonlinear Laplace and Laplace-Bessel Operator

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Abstract. In this paper, we study the solution of nonlinear equation $\triangle^k u(x) = f(x, \triangle^{k-1}u(x))$ where $\triangle^k$ is the Laplace operator iterated k-times, $x \in \mathbb{R}^n$, k is a nonnegative integer, $u(x)$ is an unknown and f is a given function. It is found that the existence of the solution $u(x)$ of such equation depending on the conditions of f and $\triangle^{k-1}u(x)$. Moreover, we study the solution of nonlinear equation $\triangle_B^k u(x) = f(x, \triangle_B^{k-1}u(x))$ where $\triangle_B^k$ is the Laplace-Bessel operator iterated k-times and $x \in \mathbb{R}_n^+$.

Keywords: Laplace-Bessel operator; Bessel ultra-hyperbolic operator; Dirac-delta distribution

1 Introduction

Soliton theory is one of the most important aspect in nonlinearity, which is widely applied in many natural sciences such as chemistry, biology, mathematics, communication and physics. For finding some new exact solutions of nonlinear equations, a wealth of some effective works have been presented [1, 2, 14]. However, a weak solution has been studied too and there are many different definitions of weak solution, appropriate for different classes of equations. One of the most important is based on the notation of distributions [3, 4, 11, 12]. In this work, we study solution of nonlinear equation $\triangle^k u(x) = f(x, \triangle^{k-1}u(x))$ and $\triangle_B^k u(x) = f(x, \triangle_B^{k-1}u(x))$ in the sense of distributions, weak solution.

R. Courant and D. Hilbert [5] have studied the nonlinear equation of the form $\triangle u(x) = f(x, u(x))$ with f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where Ω is an open set in $\mathbb{R}^n$, $\partial\Omega$ denotes the boundary of Ω and $\triangle$ is the Laplace operator, defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{1}$$

They found that the solution $u(x)$ of such equation is unique under the condition $|f(x, u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $u(x) = 0$ for all $x \in \partial\Omega$.

A. Kananthai [10] has first introduced the operator $\diamondsuit^k$ and is named the diamond operator iterated k-times and is defined by

$$\diamondsuit^k = \left[\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2\right]^k, \tag{2}$$

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$p+q=n$ and moreover, he has studied elementary solution of the n-dimensional diamond operator.

Next, A. Kananthai [7–9] has studied the convolution equation related to the diamond kernel of Marcel Riesz, the convolution of the diamond kernel of Marcel Riesz and the general solution of the equation $\diamondsuit^k u(x)=f(x)$.

Later, G. Sritanratana and A. Kananthai [17] have studied the nonlinear equation of the form

$$\diamondsuit^k u(x)=f(x,\triangle^{k-1}\square^k u(x)) \tag{3}$$

with f defined and continuous for all $x\in\Omega\cup\partial\Omega$ where $\square^k$ is the ultra-hyperbolic operator iterated k-times, defined by

$$\square^k=\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\frac{\partial^2}{\partial x_{p+2}^2}-\cdots-\frac{\partial^2}{\partial x_{p+q}^2}\right)^k. \tag{4}$$

They found that the solution $u(x)$ of (3) which is unique under the condition $|f(x,\triangle^{k-1}\square^k u(x))|\le N$ where N is a constant for all $x\in\Omega$ and the boundary condition $\triangle^{k-1}\square^k u(x)=0$ for all $x\in\partial\Omega$.

H. Yildirim et al. [18] have introduced the Bessel diamond operator iterated k-times with $x\in\mathbb{R}_n^+=\{x:x=(x_1,x_2,\ldots,x_n),x_1>0,\ldots,x_n>0\}$,

$$\diamondsuit_B^k=\left[\left(B_{x_1}+\cdots+B_{x_p}\right)^2-\left(B_{x_{p+1}}+\cdots+B_{x_{p+q}}\right)^2\right]^k \tag{5}$$

where $p+q=n$, $B_{x_i}=\frac{\partial^2}{\partial x_i^2}+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$ where $2v_i=2\beta_i+1,\beta_i>-\frac{1}{2}$[13], k is nonnegative integer and n is the dimension of $\mathbb{R}_n^+$ and studied the elementary solution of this operator. Moreover, they have studied the Fourier-Bessel transform of the elementary solution.

Next, M. Z. Sarikaya and H. Yildirim [15, 16] have studied the B-convolution of the Bessel diamond kernel of Riesz and the nonlinear equation of the form

$$\diamondsuit_B^k u(x)=f(x,\triangle_B^{k-1}\square_B^k u(x)) \tag{6}$$

with f defined and continuous for all $x\in\Omega\cup\partial\Omega$ where $\square_B^k$ is the Bessel ultra-hyperbolic operator iterated k-times, defined by

$$\square_B^k=\left(B_{x_1}+B_{x_2}+\cdots+B_{x_p}-B_{x_{p+1}}-B_{x_{p+2}}-\cdots-B_{x_{p+q}}\right)^k, \tag{7}$$

Ω is an open set in $\mathbb{R}_n^+$ and $\partial\Omega$ denotes the boundary of Ω. They found that the solution $u(x)$ of (6) which is unique under the condition $|f(x,\triangle_B^{k-1}\square_B^k u(x))|\le N$ where N is a constant for all $x\in\Omega$ and the boundary condition $\triangle_B^{k-1}\square_B^k u(x)=0$ for all $x\in\partial\Omega$.

In this work, we will study the nonlinear equation of the form

$$\triangle^k u(x)=f(x,\triangle^{k-1}u(x)) \tag{8}$$

with f defined and continuous for all $x\in\Omega\cup\partial\Omega$ where Ω is an open set in $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω. We will find the solution $u(x)$ of (8) which is unique under the condition $|f(x,\triangle^{k-1}u(x))|\le N$ where N is a constant for all $x\in\Omega$ and the boundary condition $\triangle^{k-1}u(x)=0$ for all $x\in\partial\Omega$. Moreover, we will study the nonlinear equation of the form

$$\triangle_B^k u(x)=f(x,\triangle_B^{k-1}u(x)) \tag{9}$$

with f defined and continuous for all $x\in\Omega\cup\partial\Omega$ where Ω is an open subset of $\mathbb{R}_n^+$ and $\partial\Omega$ denotes the boundary of Ω. We will find the solution $u(x)$ of (9) which is unique under the condition $|f(x,\triangle_B^{k-1}u(x))|\le M$ where M is a constant for all $x\in\Omega$ and the boundary condition $\triangle_B^{k-1}u(x)=0$ for all $x\in\partial\Omega$.

2 Preliminaries

In this section, we give some notations and definitions.

Definition 2.1 Let $x = (x_1, x_2, \dots x_n)$ be a point of $\mathbb{R}^n$ and the function $R^e_\alpha(x)$ be defined by

$$R^e_\alpha(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)}, \tag{10}$$

where $W_n(\alpha) = \pi^{n/2}2^\alpha\Gamma(\alpha/2)/\Gamma((n-\alpha)/2)$, α is a complex parameter and $|x| = (x_1^2+x_2^2+\cdots+x_n^2)^{1/2}$.

It can be shown that $R^e_{-2k}(x) = (-1)^k\triangle^k\delta$ where $\triangle^k$ is defined by (1) and δ is the Dirac-delta distribution. It follows that $R^e_0(x) = \delta$, see ([6], p.74).

Moreover, the function $(-1)^k R^e_{2k}(x)$ an elementary solution of the Laplace operator iterated k-times $\triangle^k$, that is

$$\triangle^k\left((-1)^k R^e_{2k}(x)\right) = \delta, \tag{11}$$

see ([10], Lemma 2.4).

The function $E(x) = -S_2(x)$ as defined by (14) is an elementary solution of the Laplace-Bessel operator

$$\triangle_B = \sum_{i=1}^n B_{x_i} = \sum_{i=1}^n\left(\frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\right), \tag{12}$$

that is, $\triangle_B E(x) = \delta$ where $x \in \mathbb{R}_n^+$.

The operator $\diamondsuit_B^k$ can be expressed as the product of the operators $\Box_B^k$ and $\triangle_B^k$, that is

$$\diamondsuit_B^k = \left[\left(\sum_{i=1}^p B_{x_i}\right)^2 - \left(\sum_{i=p+1}^{p+q} B_{x_i}\right)^2\right]^k = \left[\sum_{i=1}^p B_{x_i} - \sum_{i=p+1}^{p+q} B_{x_i}\right]^k\left[\sum_{i=1}^p B_{x_i} + \sum_{i=p+1}^{p+q} B_{x_i}\right]^k = \Box_B^k\triangle_B^k.$$

Denoted by T^y the generalized shift operator acting according to the law [13]

$$T_x^y\varphi(x) = C_v^*\int_0^\pi\cdots\int_0^\pi\varphi\left(\sqrt{x_1^2+y_1^2-2x_1y_1\cos\theta_1},\dots,\sqrt{x_n^2+y_n^2-2x_ny_n\cos\theta_n}\right)$$
$$\times\left(\prod_{i=1}^n\sin^{2v_i-1}\theta_i\right)d\theta_1\cdots d\theta_n,$$

where $x, y \in \mathbb{R}_n^+$, $C_v^* = \prod_{i=1}^n\frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator [13]

$$\frac{d^2U}{dx^2} + \frac{2v}{x}\frac{dU}{dx} = \frac{d^2U}{dy^2} + \frac{2v}{y}\frac{dU}{dy}, \qquad U(x,0) = f(x), \qquad U_y(x,0) = 0.$$

The convolution operator determined by the T^y is as follows:

$$(f*\varphi)(y) = \int_{\mathbb{R}_n^+} f(y)T_x^y\varphi(x)\left(\prod_{i=1}^n y_i^{2v_i}\right)dy. \tag{13}$$

Convolution (13) known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator.

(a) $T_x^y\cdot 1 = 1$;

(b) $T_x^0\cdot f(x) = f(x)$;

(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function for all $x \in R_n^+$ and

$$\int_{\mathbb{R}_n^+} |f(x)| \left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx < \infty$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy.$$

(d) From (c), we have the following equality for $g(x) = 1$.

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy.$$

(e) $(f * g)(x) = (g * f)(x)$.

The proof of the following lemmas can be seen in [18].

Lemma 2.1 *Given the equation $\triangle_B u(x) = \delta$ for $x \in \mathbb{R}_n^+$, where $\triangle_B$ is the Laplace-Bessel operator defined by (12). Then $E(x) = -S_2(x)$ is an elementary solution of the operator $\triangle_B$ where*

$$E(x) = -S_2(x) = -\frac{2^{n+2|v|-4}\Gamma\left(\frac{n+2|v|-2}{2}\right)}{\prod_{i=1}^{n} 2^{v_i-\frac{1}{2}}\Gamma\left(v_i+\frac{1}{2}\right)} |x|^{2-n-2|v|}. \tag{14}$$

Lemma 2.2 *Given the equation $\triangle_B^k u(x) = \delta$ for $x \in \mathbb{R}_n^+$, where $\triangle_B^k$ is the Laplace-Bessel operator iterated k-times defined by (12). Then $u(x) = (-1)^k S_{2k}(x)$ is an elementary solution of the operator $\triangle_B^k$ where*

$$S_{2k}(x) = \frac{2^{n+2|v|-4k}\Gamma\left(\frac{n+2|v|-2k}{2}\right)}{\prod_{i=1}^{n} 2^{v_i-\frac{1}{2}}\Gamma\left(v_i+\frac{1}{2}\right)\Gamma(k)} |x|^{2k-n-2|v|}. \tag{15}$$

The proof of the following lemmas can be seen in [17].

Lemma 2.3 *Given the equation*

$$\triangle^k u(x) = 0, \tag{16}$$

where $\triangle$ is defined by (1) and $x \in \mathbb{R}^n$, then $u(x) = (-1)^{k-1}(R^e_{2(k-1)}(x))^{(m)}$ as a solution of (16) where m is a nonnegative integer with $m = (n-4)/2, n \geq 4$ and n is even and $(R^e_{2(k-1)}(x))^{(m)}$ is a function defined by (10) with m derivatives and $\alpha = 2(k-1)$.

Lemma 2.4 *Given the equation*

$$\triangle u(x) = f(x, u(x)), \tag{17}$$

where f is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open set in $\mathbb{R}^n$ and $\partial\Omega$ is the boundary of Ω, assume that f is bounded, that is $|f(x,u)| \leq N$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ as a unique solution of (17).

The proof of the following lemmas can be seen in [16].

Lemma 2.5 *Given the equation*

$$\triangle_B^k u(x) = 0, \tag{18}$$

where $\triangle_B$ is defined by (12) and $x \in \mathbb{R}_n^+$, then $u(x) = (-1)^{k-1}(S_{2(k-1)}(x))^{(m)}$ as a solution of (18) where m is a nonnegative integer with $m = (n+2|v|-4)/2, n+2|v| \geq 4$ and n is even and $(S_{2(k-1)}(x))^{(m)}$ is a function defined by (15) with m derivatives.

Lemma 2.6 *Given the equation*

$$\triangle_B u(x) = f(x, u(x)), \tag{19}$$

where f is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open set in $\mathbb{R}_n^+$ and $\partial\Omega$ is the boundary of Ω, assume that f is bounded, that is $|f(x,u)| \leq M$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ as a unique solution of (19).

3 Main Results

In this section, we will state our main results and give their proofs.

Theorem 3.1 *Consider the nonlinear equation*

$$\triangle^k u(x) = f(x, \triangle^{k-1} u(x)), \tag{20}$$

where $\triangle^k$ is the Laplace operator iterated k-times, defined by (1), let f be defined and have continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, where Ω is an open set in $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω and n is even with $n \geq 4$. Let f be a bounded function, that is for all $x \in \Omega$,

$$|f(x, \triangle^{k-1} u(x))| \leq N \tag{21}$$

and the boundary condition for $x \in \partial\Omega$ be

$$\triangle^{k-1} u(x) = 0. \tag{22}$$

Then we obtain

$$u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * W(x) \tag{23}$$

as a solution of (20) with the boundary condition

$$u(x) = (-1)^{k-2} (R^e_{2(k-2)}(x))^{(m)} \quad \text{for } x \in \partial\Omega$$

where $m = (n-4)/2, k = 2, 3, \ldots$ and $W(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$, $R^e_{2(k-2)}(x)$ is given by (10) with $\alpha = 2(k-2)$.

Proof We have

$$\triangle^k u(x) = \triangle(\triangle^{k-1} u(x)) = f(x, \triangle^{k-1} u(x)). \tag{24}$$

Since $u(x)$ has continuous derivatives up to order $2k$ for $k = 1, 2, 3, \ldots$, we can assume for all $x \in \Omega$

$$\triangle^{k-1} u(x) = W(x). \tag{25}$$

Thus, (24) can be written in the form

$$\triangle^k u(x) = \triangle W(x) = f(x, W(x)), \tag{26}$$

by (21), for all $x \in \Omega$

$$|f(x, W(x))| \leq N \tag{27}$$

and by (22), $W(x) = 0$ or for all $x \in \partial\Omega$

$$\triangle^{k-1} u(x)) = 0. \tag{28}$$

Thus by Lemma 2.4 there exists a unique solution $W(x)$ of (26) which satisfies (27). Convolving both sides of (25) by $(-1)^{k-1} R^e_{2(k-1)}(x)$, we obtain

$$(-1)^{k-1} R^e_{2(k-1)}(x) * \triangle^{k-1} u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * W(x),$$

or

$$\triangle^{k-1} \left((-1)^{k-1} R^e_{2(k-1)}(x) \right) * u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * W(x),$$

or

$$\delta * u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * W(x).$$

Thus

$$u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * W(x) \tag{29}$$

as required. We consider for $x \in \partial\Omega$

$$\triangle^{k-1} u(x) = 0.$$

By Lemma 2.3, we have

$$u(x) = (-1)^{k-2} (R^e_{2(k-2)}(x))^{(m)}$$

where $m = (n-4)/2, n \geq 4$ and n is even and $k = 2, 3, \ldots$.

This complete the proof. □

Theorem 3.2 *Consider the nonlinear equation*

$$\triangle_B^k u(x) = f(x, \triangle_B^{k-1} u(x)), \tag{30}$$

where $\triangle_B^k$ *is the Laplace-Bessel operator iterated* k*-times, defined by (12), let* f *be defined and have continuous first derivatives for all* $x \in \Omega \cup \partial\Omega$*, where* Ω *is an open set in* $\mathbb{R}_n^+$ *and* $\partial\Omega$ *denotes the boundary of* Ω *and* n *is even with* $n + 2|v| \geq 4$*. Let* f *be a bounded function, that is for all* $x \in \Omega$*,*

$$|f(x, \triangle_B^{k-1} u(x))| \leq M \tag{31}$$

and the boundary condition for $x \in \partial\Omega$ *be*

$$\triangle_B^{k-1} u(x) = 0. \tag{32}$$

Then we obtain

$$u(x) = (-1)^{k-1} S_{2(k-1)}(x) * W(x) \tag{33}$$

as a solution of (30) with the boundary condition

$$u(x) = (-1)^{k-2} (S_{2(k-2)}(x))^{(m)} \quad \text{for } x \in \partial\Omega$$

where $m = (n + 2|v| - 4)/2, k = 2, 3, \ldots$ *and* $W(x)$ *is a continuous function for* $x \in \Omega \cup \partial\Omega$*,* $S_{2(k-2)}(x)$ *is given by (15).*

The proof of this Theorem is similar to the proof of Theorem 3.1, by using Lemma 2.5, 2.6 and B-convolution.

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