ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.8(2009) No.4,pp.469-472

On Fractional Order Quantum Mechanics

A.M. Shahin, E.Ahmed *, Yassmin A.Omar
Department of Mathematics, Faculty of Science, Mansoura University
Mansoura 35516, Egypt
(Received 28 July 2009, accepted 15 September 2009)

Abstract: It is argued that fractional order (FO) calculus is more suitable to describe fractal systems. Motivated by the fractal space time theory some fractional generalizations of Scrodinger and Klein-Gordon equations are given. Fractional order calculus naturally includes nonlocality which is a known property of quantum systems.

Keywords:fractional calculus; fractal space time

1 Introduction

It is argued that fractional order (FO) calculus is more suitable to describe complex and fractal systems. Recently fractal space time has been proposed to solve some of the remaining problems in quantum mechanics and quantum field theory. It assumes that at small scale, space time is fractal. Hence we propose that such systems should be described by fractional order systems. In sec.2 fractional order calculus is reviewed. In sec.3 fractional order quantum mechanics is studied. Some conclusions are given in sec.4.

2 Fractional Equations

Caputo's definition for derivative of order $0 < \alpha \le 1$ is given by

$$D^{\alpha} f(t) = (1/\Gamma(1-\alpha)) \int_0^t [df(s)/ds]/(t-s)^{\alpha}$$

Consider the following evolution equation [1]

$$df(t)/dt = -\lambda^2 \int_0^t k(t-t')f(t')dt' \quad (1)$$

If the system has no memory then $k(t-t') = \delta(t-t')$ and one gets $f(t) = f_0 \exp(\lambda^2 t)$. If the system has an ideal memory then $k(t-t') = \left\{1 \text{ if } t \ge t', 0 \text{ if } t < t'\right\}$ hence $f \approx f_0 \cos \lambda t$. Using Laplace transform $L[f] = \int_0^{\infty} f(t) \exp(-st)dt$ one gets $L[f] = 1$ if there is no memory and $1/s$ if there is ideal memory hence the case of non-ideal memory is expected to be given by $L[f] = 1/s^{\alpha}, 0 < \alpha < 1$. In this case equation (1) becomes

$$df(t)/dt = \int_0^t (t-t')f(t')dt'/\Gamma(\alpha) \quad (2)$$

*__Corresponding author__. *E-mail address*: magd45@yahoo.com

where $\Gamma(\alpha)$ is the Gamma function. This system has the following solution $f(t) = f_0 E_{\alpha+1}(-\lambda^2 t^{\alpha+1})$, where $E_\alpha(z)$ is the Mittag Leffler function given by

$$E_\alpha(z) = \sum_{k=0}^{\infty} z^k / \Gamma(ak+1)$$

Following a similar procedure to study a random process with memory, one obtains the following fractional evolution equation

$$\partial^{\alpha+1} P(x,t)/\partial t^{\alpha+1} = \sum_{k=0}^{\infty} (-1)^n / n! \partial^n \left[K_n(x) P(x,t)\right]/\partial x^n, \qquad 0 < \alpha < 1 \tag{3}$$

where P(x,t) is a measure of the probability to find a particle at time t at position x.

We expect that (3) will be relevant to many complex adaptive systems and to systems where fractal structures are relevant since it is argued that there is a relevance between fractals and fractional differentiation [2,3].

For the case of fractional diffusion equation the results are

$$\begin{array}{l} \partial^{\alpha+1} P(x,t)/\partial t^{\alpha+1} = D\partial^2 P(x,t)/\partial x^2, \ \text{P(x,0)} = \delta(x), \partial P(x,0)/\partial t = 0 \Rightarrow \\ P = (1/(2\sqrt{D} t^\beta) M(|x| / \sqrt{D} t^\beta; \beta), \ \beta = (\alpha+1)/2 \\ M(z;\beta) = \sum\limits_{n=0}^{\infty} \left[(-1)^n z^n / \{n! \Gamma(-\beta n + 1 - \beta)\}\right] \end{array} \tag{4}$$

For the case of no memory $\alpha = 0 \Rightarrow M(z; 1/2) = \exp(-z^2/4)$.

3 Proposed fractional equations in quantum systems

When one studies quantum mechanics one feels a sudden change from classical mechanics and old quantum mechanics on the one hand to modern quantum mechanics (beginning with Schrodinger equation) on the other side. By this we mean that suddenly we leave the familiar concepts of position, momentum, energy, etc... and we replace them by hermitian operators which eventually gives Schrodinger equation. Can the latter equation be derived from the familiar Euler-Lagrange equations?. The answer is yes according to the scale relativity theory [4-7]. According to this theory, both space-time and scale are considered. The importance of scales is that observations and possibly physical laws depend on the scale (resolution of the experiment).

Moreover in this theory it is argued that space-time is fractal (continuous but nowhere differentiable). Accordingly Nottale has shown (among several results) that Schrodinger equation is equivalent to the familiar Euler-Lagrange equations on a fractal. Thus the gap between classical and quantum mechanics have been removed.

Scale relativity theory has two main assumptions: The first is that one should study scale (Δ) in addition to space and time (x,y,z,t). The second assumption is that space time , at least in the microscale, is a fractal in the general sense i.e. it is continuous but not differentiable. The first assumption agrees with the observation that different theories are used to describe phenomena at different scales e.g. quantum mechanics is used in the micro-scale while classical mechanics is used in the intermediate ones. Planck's time and length play a role for scales similar to light speed in special relativity. The second assumption implies that any displacement $dX_\pm$ should be decomposed into left and right which are given by

$$dX_s = < dX_\pm > + d\zeta_\pm, \ < d\zeta_\pm > = 0, \ < d\zeta_\pm, d\zeta_\pm > \propto (dt)^{2/D}$$

where D is the space time fractal dimension . Applying Ito calculus procedure, it has been shown [4,5] that Schrodinger equation is equivalent to Euler -Lagrange equation on a fractal.

There is a relation between chaos and the non-differentiability of space time since if a straight line $x = 0, y = 0, z = at$, a constant is perturbed to

$$x = \epsilon_1(1 + \exp(t/\lambda)), \quad y = \epsilon_2(1 + \exp(t/\lambda)), \quad z = at + \epsilon_3(1 + \exp(t/\lambda))$$

where λ is Lyapunov exponent of the system then it is direct to see (by eliminating the time) that dz/dx is not defined as $x \to 0$. This may be related to the existence, in many phenomena, of a horizon of predictability.

Nottale [4,5] has applied the Schrodinger type equation resulting from scale relativity to gravity and obtained

$$D^2 \bigtriangledown^2 \psi + iD\partial\psi/\partial t + GM\psi/r = 0$$

$|\psi|^2$ is a measure for making structures e.g. planets, stars ,galaxies, etc and used it to describe some astrophysical phenomena.

We think that the following points makes scale relativity interesting and deserves further research:

1) The dependence on scale is an observed phenomena.

2) It relates classical and quantum mechanics in a natural way by proving that Schrodinger equation is equivalent to Euler -Lagrange equation on a fractal.

3) Since earthquakes mostly occur on fractal faults, applying scale relativity to the earthquake phenomena is an interesting application [11].

4) Since it has been found that fractional calculus is more natural to apply on fractals [2,3] generalizing scale relativity equations to fractional order should be done. It is interesting to know that several attempts have been done to generalize Schrodinger equation to fractional order as will be seen next section.. According to scale relativity theory, fractals are expected to be a relevant concept in quantum systems. It is known that the fractional equations (FE) are the natural tools to study fractal systems. Therefore it is relevant to attempt to use FE to generalize both Schrodinger and Klein-Gordon equations.

In the next section it will be shown that FE are also relevant to systems with memory which typically are open and dissipative systems and to complex systems [8,9].

Motivated by the previous two sections we propose the following fractional equation to describe quantum systems:

$$e^{i\pi\beta}\hbar\partial^{2\beta}\psi/\partial t^{2\beta} = (\hbar^2/2m)(-\Delta)^{\gamma}\psi + V\psi \tag{5}$$

where $\beta = (1+\alpha)/2, \alpha$ is in (4), ψ is the wave function of the system, V is the potential and Δ is Laplace operator.

It is clear that (5) reduces to the familiar Schrodinger equation if $(\alpha \to 0, \gamma \to 1)$ and is equivalent to Klein-Gordon equation as $(\alpha \to 1, \gamma \to 1)$.

Recently [10] the fractional potential well has been solved. It is given by (up to constants) (5) with potential function $V(x) = \{0 \text{ if } 0 < x < a, \infty \text{ otherwise}\}$.

Thus using separation of variables one gets

$$\psi = f(t)\sin(n\pi x/a), \quad n = 1, 2, 3, \cdots \tag{6}$$

It is straightforward to see that $f(t) = E_{2\beta}(\omega_n(-it)^{2\beta})$ where $E_{2\beta}$ is Mittag-Leffler function and ω_n are parameters. Using Naber's formula

$$E_\nu(\omega(-it)^\nu) = (1/\nu)\left\{\exp(-i\omega^{1/\nu}t) - \nu F_\nu((-i\omega)^\nu, t)\right\} \tag{7}$$

where F decays monotonically with time. Thus as time increases the first term in (7) becomes dominant. Consequently the energy of the nth level is proportional to $\epsilon_n \propto n^{2\beta}$. Similar argument leads us to expect that for the fractional hydrogen atom the energy of the nth level is $\epsilon_n \propto -1/n^{2\beta}$ Thus the fractional approach affects the energy levels but it preserves quantization.

Another application of fractal space time is to model earthquakes [11,12] since it is known that earthquakes occur on faults which are mostly fractals. Generalizing to fractional models for earthquakes has already been done [12].

Recently [5] a quantum space time theory has been proposed and was successful in deriving mass spectrum of QED and QCD particles. It also depends on the assumption that at micro scale, space time has a hierarchical Cantor like fractal structure. Therefore we expect FE to be relevant to this theory. It is also expected that the intrinsic non locality of fractional calculus may explain the entanglement problem in quantum mechanics.

4 Conclusions

Complex systems (CS) are frequent in nature. Mathematical methods can be helpful in understanding them. Motivated by the properties of CS we concluded that:

i) Fractional order systems are more suitable for describing CS for the following reasons: First they are more natural in describing fractal systems. Second they are more natural in describing systems with memory and delay. Third they are more natural in describing non-local systems. This may have an impact on the entanglement problem in quantum mechanics.

ii) Motivated by the theory of fractal space time, fractional order generalization for Schrodinger equations are presented.

iii) Fractional order automatically include nonlocality which is a known property of quantum systems.

It is interesting that the bright idea of Nottale, Ord and El-Naschie namely fractal space time offers a unifying scheme for such diverse fields as quantum mechanics, earthquakes and fractional order equations. Also it is interesting to see the wide range of applications of nonlinear systems e.g. algebra [13], game theory [14], economics [15] an addition to physics [16].

References

[1] A.A.Stanislavsky: Memory effects and macroscopic manifestation of randomness. *Phys. Rev.E.* 61:4752-4759 (2000)

[2] A.Rocco, B.J.West: Fractional Calculus and the Evolution of Fractal Phenomena. *Physica A*. 265:535-546 (1999)

[3] M.A.Zahran, E.M.Abulwafa, S.A.Elwakil: The fractional Fokker-Planck equation on comb-like model. *Physica A*. 323:237-248 (2003)

[4] L.Nottale: Fractal space-time and microphysics. *World Sci.* Singapore.(1993)

[5] L.Nottale: Scale relativity and quantization of the universe. *Astron.Astrophys.* 327:867-887 (1997)

[6] M.S.El Naschie: A note on Heisenberg's uncertainty principle and Cantorian space-time. *Chaos, Solitons and Fractals*. 2:437-439 (1992)

[7] G. N. Ord: Fractal space-time: a geometric analogue of relativistic quantum mechanics. *J.Phys.A*. 16: 1869-1884 (1983)

[8] N. Boccara: Modeling complex systems. *Springer Publ.* Berlin. (2004)

[9] W. Chen: Soft matter and fractional mathematics. cond-mat/0405345(2004)

[10] M. Naber: Time fractional Schrodinger equation: *J.Math.Phys.* 45:3339-3354 (2004)

[11] E. Ahmed, M.H.Mousa: On the application of the fractal space-time approach to dissipative chaos. *Chaos, Solitons and Fractals*. 8:805-808 (1997)

[12] A.E.M. ElMisiery, E.Ahmed: On a fractional model for Earthquakes. *Appl. Math. Comp.* 178:207-212 (2006)

[13] E.Ahmed, A.S.Hegazi, N.F.Abdo: On Deformed algebra in Particle Physics. *Int. Jour. Nonlin. Sci.* 7:493-495 (2009)

[14] E.Ahmed, H.A.Abdusalam: On Dynamical games. *Int. Jour. Nonlin. Sci.* 6:59-63 (2008)

[15] E.Elabbasy, H.Agiga, A.Elsadany, H.El-Metwally: The dynamics of tripoly games with hetrogenous players. *Int. Jour. Nonlin. Sci.* 3:83-90 (2007)

[16] S.El-Wakil, E. Aboulwafa, M. Abdou: New periodic wave solutions to nonlinear evolution equations arising in physics. *Int. Jour. Nonlin. Sci.* 7:75-83 (2009)