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New Exact Traveling Wave Solutions of Nonlinear Evolution Equations

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Abstract: In this work, we employ the $(\frac{G'}{G})$-expansion method to construct the traveling wave solutions involving parameters of nonlinear evolution equations. The traveling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions. The work shows the power of the proposed method and the variety of obtained solutions.

Keywords: $(\frac{G'}{G})$-expansion method; the (2+1)-dimensional Nizhnik-Novikov-Veselov equations; the Jimbo-Miwa equation; traveling wave solutions

1 Introduction

Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, fluid dynamics, solid state physics, optical fibers, chemical kinematics and many others. Nonlinear wave phenomena of dispersion,dissipation, diffusion, reaction and convention are very important in nonlinear sciences. During the past several decades, many powerful and efficient methods have been proposed to obtain the exact traveling and solitary wave solutions of nonlinear evolution equations (NLEEs), such as inverse scattering method [1], Hirota's bilinear method [2], Backlund transformation [3], homogenous balance method [4], tanh function method [5], Sine-Cosine function method [6, 7], extended tanh function method [8], exp-function method [9, 10], new auxiliary equation method [11] and so on.

Recently, Wang *et al.* [12] introduced a new direct method called $(\frac{G'}{G})$-expansion method for a reliable treatment of the NLEEs. The useful $(\frac{G'}{G})$-expansion method is widely used by many authors [13, 14] and Ref,s there in. More recently, this method has been generalized by Zhang *et al.* [15, 16] to obtain non-traveling wave solutions and coefficient function solutions. Later, Zhang et al. [17] further extend the method to solve an evolution equation with variable coefficients. Also, Zhang et al. [18] devised an algorithm for using the method to solve nonlinear differential-difference equations. Then, Bin and Chao [19] modified the method to derive traveling wave solutions for Witham-Broer-kaup-Like equations.

Our aim in this work is to present the solutions of higher-dimensional nonlinear partial differential equations, especially in (2+1) and (3+1)-dimensional as an application of the $(\frac{G'}{G})$-expansion method. The rest of the paper is organized as follows. In section 2, we briefly describe the $(\frac{G'}{G})$-expansion method. In section 3, we apply the method to the (2+1)-dimensional Nizhnik-Novikov-Veselov equations (NNVEs) and (3+1)-dimensional Jimbo-Miwa equation respectively. In section 4, some conclusions are given.

2 The $(\frac{G'}{G})$- expansion method: [12]

Suppose we have a nonlinear partial differential equation for $u(x,\ y,\ t)$ in the form

$$P(u,\ u_x,\ u_y,\ u_t,\ u_{xx},\ u_{xy},\ u_{xt}, u_{tt},\ u_{yy},\ u_{yt},\ u_{zx},\ u_{zy},\ \cdots) = 0, \tag{1}$$

where P is a polynomial in its arguments, which includes nonlinear terms and highest order derivatives.
Step 1. Seek traveling wave solutions of Eq. (1) by taking $u(x,\ y,\ t) = u(\xi),\ \xi = \alpha x + \beta y - ct$, and

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transform Eq. (1) to the ordinary differential equation (ODE)

$$Q(u,\ \alpha u',\ \beta u',\ -cu',\ \alpha^2 u'',\ \alpha\beta u'',\ -\alpha c u'',\ c^2 u'',\ \cdots) = 0, \tag{2}$$

where prime denotes the derivatives with respect to ξ.
Step 2. If possible, integrate Eq. (2) term by term one or more times yields constant(s) of integration. For simplicity the integration constant(s) can be set to zero .
Step 3. Suppose that the solution $u(\xi)$ of ODE (2) can be expressed as a finite series in the form

$$u(\xi) = \sum_{i=0}^{n} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{3}$$

where a_i are real constants with $a_n \neq 0$ to be determined, n is a positive integer, which is determined by the homogeneous balancing method and the function G(ξ) is the general solution of the auxiliary linear second order ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{4}$$

where λ and μ are real constants to be determined later.
Step 4. Substituting (3) together with (4) into Eq. (2) yields an algebraic equation involving powers of $(\frac{G'}{G})$. Equating the coefficients of each power of $(\frac{G'}{G})$ to zero, to obtain a system of algebraic equations for $a_i,\ \lambda,\ \mu$ and c. Then, to determine these constants we solve the system with the aid of a computer algebra system, such as Maple. Since the solutions of Eq. (4) have been well known for us depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, the exact solutions of given Eq. (1) can be obtained.

3 Applications

3.1 (2+1)-dimensional Nizhnik-Novikov-Voselov equations (NNVEs)

Consider the following (2+1)-dimensional NNVEs [20, 21] of the form

$$\begin{aligned} u_t + ku_{xxx} + ru_{yyy} + su_x + qu_y &= 3k(uv)_x + 3r(uw)_y, \\ u_x &= v_y, \\ u_y &= w_x, \end{aligned} \tag{5}$$

where $k,\ r,\ s$ and q are arbitrary constants.
In the past years, many authors have studied the NNVEs. For example, Boiti et al. [22] solved the NNVEs by the inverse scattering transformation. Tagami [23] obtained the soliton-like solutions by means of the Backlund transformation method. Ren [24] and Xia [25] obtained the Jacobi Elliptic Function solutin of the NNVEs by Sinh-Cosh method. Lou [26] analyzed the coherent structures of the NNVEs by separation of variable approach. Also, Lou [27] obtained the multi soliton solutions of the NNVEs by the standard truncated Painleve' analysis approach. Yusufoglu and Bekir [28] obtained the exact traveling wave solutions by the tanh method and the relationships between the NNVE and the Volterra hierarchy are studied by Vekslerchik [29].
Now, In order to seek the traveling wave solution of Eq. (5), we make the following transformation:

$$u(x,y,t) = u(\xi),\ v(x,y,t) = v(\xi),\ w(x,y,t) = w(\xi),\ \xi = \alpha x + \beta y - ct, \tag{6}$$

where $\alpha,\ \beta$ and c are arbitrary constants, then after integration the obtained (ODE) once, we get

$$\begin{gathered} (k\alpha^3 + r\beta^3)u'' + (s\alpha + q\beta - c)u - 3k\alpha(uv) - 3r\beta(uw) + c_1 = 0, \\ \alpha u - \beta v + c_2 = 0, \\ \beta u - \alpha w + c_3 = 0, \end{gathered} \tag{7}$$

where $c_1,\ c_2$ and c_3 are integration constants to be determined later.
Now, we make the ansatz (3) for the solution of Eq. (7). Balancing the term u'' with uv, term u'' with uw in Eq. $(7)_1$ and the term u with v in Eq. $(7)_2$ or the term u with w in Eq. $(7)_3$ yields the leading orders $m=n=l=2$, therefore, we can write the solution of Eq. (7) in the form

$$u(\xi) = \alpha_2\left(\frac{G'}{G}\right)^2+\alpha_1\left(\frac{G'}{G}\right)+\alpha_0,\quad \alpha_2\neq 0, \tag{8}$$

$$v(\xi) = \beta_2\left(\frac{G'}{G}\right)^2+\beta_1\left(\frac{G'}{G}\right)+\beta_0,\quad \beta_2\neq 0, \tag{9}$$

$$w(\xi) = \delta_2\left(\frac{G'}{G}\right)^2+\left(\delta_1\frac{G'}{G}\right)+\delta_0,\quad \delta_2\neq 0. \tag{10}$$

Substituting (8)–(10) together with (4) into Eq. (7), collecting all terms with the same powers of $(\frac{G'}{G})$ and setting each coefficients to zero, we obtain the following system of algebraic equations:

$$\begin{aligned}
c_1+(k\alpha^3+r\beta^3)(2\mu^2\alpha_2+\lambda\mu\alpha_1)+(s\alpha+q\beta-c)\alpha_0-3k\alpha\alpha_0\beta_0-3r\beta\alpha_0\delta_0=0,\\
(k\alpha^3+r\beta^3)(6\mu\alpha_2\lambda+2\mu\alpha_1+\lambda^2\alpha_1)+(s\alpha+q\beta-c)\alpha_1-3k\alpha(\alpha_1\beta_0+\alpha_0\beta_1)-\\
-3r\beta(\alpha_1\delta_0+\alpha_0\delta_1)=0,\\
(k\alpha^3+r\beta^3)(8\mu\alpha_2+3\lambda\alpha_1+4\lambda^2\alpha_2)+(s\alpha+q\beta-c)\alpha_2-3k\alpha(\alpha_2\beta_0+\alpha_1\beta_1+\alpha_0\beta_2)-\\
-3r\beta(\alpha_2\delta_0+\alpha_1\delta_1+\alpha_0\delta_2)=0,\\
(k\alpha^3+r\beta^3)(10\alpha_2\lambda+2\alpha_1)-3k\alpha(\alpha_1\beta_2+\alpha_2\beta_1)-3r\beta(\alpha_1\delta_2+\alpha_2\delta_1)=0,\\
(6k\alpha^3+r\beta^3)\alpha_2-3k\alpha\alpha_2\beta_2-3r\beta\alpha_2\delta_2=0,\\
c_2+\alpha\alpha_0-\beta\beta_0=0,\\
\alpha\alpha_1-\beta\beta_1=0,\\
\alpha\alpha_2-\beta\beta_2=0,\\
c_3+\beta\alpha_0-\alpha\delta_0=0,\\
\beta\alpha_1-\alpha\delta_1=0,\\
\beta\alpha_2-\alpha\delta_2=0.
\end{aligned}$$

Solving the above system with the aid of Maple 10, we have the following set of solutions:

$$\alpha_2=2\alpha\beta,\ \ \alpha_1=2\lambda\alpha\beta,\ \ \beta_2=2\alpha^2,\ \ \beta_1=2\lambda\alpha^2,\ \ \delta_2=2\beta^2,\ \ \delta_1=2\lambda\beta^2,$$

$$\begin{aligned}
c_1=-\frac{1}{\alpha\beta}\big[3k\alpha^3\alpha_0^2-\lambda^2k\alpha^4\beta\alpha_0-8\mu k\alpha^4\beta\alpha_0-\lambda^2r\alpha\beta^4\alpha_0-8\mu r\alpha\beta^4\alpha_0+3r\beta^3\alpha_0^2+4\mu^2k\alpha^5\beta^2+\\
+4r\mu^2\alpha^2\beta^5+2\lambda^2\mu r\alpha^2\beta^5\big],
\end{aligned} \tag{11}$$

$$c_2=-\alpha\alpha_0+\beta\beta_0, c_3=-\beta\alpha_0+\alpha\delta_0,$$

$$\begin{aligned}
c=\frac{1}{\alpha\beta}[-3k\alpha^3\alpha_0+\lambda^2k\alpha^4\beta+8\mu k\alpha^4\beta+\lambda^2r\alpha\beta^4+8\mu r\alpha\beta^4+\beta\delta\alpha^2+q\alpha\beta^2-3k\beta\beta_0\alpha^2-3r\alpha\beta^2\delta_0-\\
-3r\beta^3\alpha_0].
\end{aligned}$$

Now, Substituting (11) together with the solution of the Eq. (4) into (8)–(10), we obtain three types of traveling wave solutions of Eq. (5) as follows:
when $\lambda^2-4\mu>0$,

$$u(\xi)=\frac{\alpha\beta}{2}(\lambda^2-4\mu)\left(\frac{A_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+A_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{A_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+A_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2-\frac{1}{2}\lambda^2\alpha\beta+\alpha_0, \tag{12}$$

$$v(\xi)=\frac{\alpha^2}{2}(\lambda^2-4\mu)\left(\frac{A_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+A_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{A_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+A_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2-\frac{\alpha^2\lambda^2}{2}+\beta_0, \tag{13}$$

$$w(\xi)=\frac{\beta^2}{2}(\lambda^2-4\mu)\Big(\frac{A_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+A_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{A_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+A_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\Big)^2-\frac{\beta^2\lambda^2}{2}+\delta_0, \quad (14)$$

where A_1 and A_2 are arbitrary constants.
when $\lambda^2-4\mu<0$,

$$u(\xi)=\frac{\alpha\beta}{2}(4\mu-\lambda^2)\Big(\frac{-A_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+A_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{A_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+A_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\Big)^2-\frac{1}{2}\lambda^2\alpha\beta+\alpha_0, \quad (15)$$

$$v(\xi)=\frac{\alpha^2}{2}(4\mu-\lambda^2)\Big(\frac{-A_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+A_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{A_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+A_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\Big)^2-\frac{\alpha^2\lambda^2}{2}+\beta_0, \quad (16)$$

$$w(\xi)=\frac{\beta^2}{2}(4\mu-\lambda^2)\Big(\frac{-A_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+A_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{A_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+A_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\Big)^2-\frac{\beta^2\lambda^2}{2}+\delta_0, \quad (17)$$

when $\lambda^2-4\mu=0$,

$$u(\xi)=2\alpha\beta\Big(\frac{A_2}{A_1+A_2\xi}\Big)^2-\frac{1}{2}\lambda^2\alpha\beta+\alpha_0, \quad (18)$$

$$v(\xi)=2\alpha^2\Big(\frac{A_2}{A_1+A_2\xi}\Big)^2-\frac{\alpha^2\lambda^2}{2}+\beta_0, \quad (19)$$

$$w(\xi)=2\beta^2\Big(\frac{A_2}{A_1+A_2\xi}\Big)^2-\frac{\beta^2\lambda^2}{2}+\delta_0. \quad (20)$$

In Particular, If we take $A_1=0$, $A_2\neq 0, \lambda>0$ and $\mu=0$ then the solutions (12)–(14) becomes

$$u(\xi)=-\frac{\alpha\beta}{2}\lambda^2 sech^2\frac{\lambda}{2}\xi+\alpha_0, \quad (21)$$

$$v(\xi)=-\frac{\alpha^2\lambda^2}{2}\lambda^2 sech^2\frac{\lambda}{2}\xi+\beta_0, \quad (22)$$

$$w(\xi)=-\frac{\beta^2\lambda^2}{2}\lambda^2 sech^2\frac{\lambda}{2}\xi+\delta_0, \quad (23)$$

which are the solitary wave solutions of the (2+1)-dimensional NNVEs. Here we would like to mention that the rational solutions (18)–(20) do not appear in literature. Furthermore, it is possible to recover the same solutions obtained in [28] by taking $\lambda=2$ in (21)–(23).

3.2 (3+1)-dimensional Jimbo-Miwa equation

Next, we consider the (3+1)-dimensional Jimbo-Miwa equation [30–33] of the form

$$u_{xxxy}+3u_yu_{xx}+3u_xu_{xy}+2u_{yt}-3u_{xz}=0. \quad (24)$$

Jimbo-Miwa equation is firstly investigated by Jimbo and Miwa [34] and then it is studied by several authors regarding its solutions [35], symmetries [36] and integrability properties [37]. Senthilelan [38] studied the traveling wave solutions of (3+1)-dimensional Jimbo-Miwa equation by means of the Homogenous Balance Method. In a recent study, Wazwaz [39, 40] employed the Hirota's bilinear method to the Jimbo-Miwa equation and showed that it is completely integrable and admits multiple soliton solutions of any order. He also successfully studied one soliton solutions to Eq. (24) by means of the tanh-coth method.
Now to seek the traveling wave solutions of Eq. (24) we make the following transformation:

$$u(x,y,z,t)=u(\xi),\ \xi=\alpha x+\beta y+\gamma z-ct, \quad (25)$$

where α,β,γ and c are constants, then after integration the obtained (ODE) once, we get,

$$\alpha^3\beta u'''+3\beta\alpha^2(u')^2-(2\beta c+3\alpha\gamma)u'+c_1=0, \quad (26)$$

where c_1 is an integration constat, to be determined later.
Now, we make the ansatz (3) for the solution of Eq. (26). Balancing the term u''' with $(u')^2$ in Eq. (26) yields the leading order $n = 1$. Therefore, we can write the solution of Eq. (26) in the form

$$u(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0. \tag{27}$$

Substituting (27) together with (4) into Eq. (26) collecting all terms with the same powers of $(\frac{G'}{G})$ and setting the coefficients to zero, we obtain the following system of algebraic equations:

$$\begin{aligned}
c_1 - \alpha^3\beta\lambda^2\mu\alpha_1 - 2\mu^2\alpha^3\beta\alpha_1 + 3\beta\alpha^2\mu^2\alpha_1^2 + (2\beta c + 3\alpha\gamma)\mu\alpha_1 &= 0,\\
-\alpha^3\beta\lambda^3\alpha_1 - 8\lambda\mu\alpha^3\beta\alpha_1 + 6\beta\alpha^2\mu\lambda\alpha_1^2 + (2\beta c + 3\alpha\gamma)\lambda\alpha_1 &= 0,\\
-7\alpha^3\beta\lambda^2\alpha_1 - 8\mu\alpha^3\beta\alpha_1 + 3\beta\alpha^2(\lambda^2\alpha_1^2 + 2\mu\alpha_1^2) + (2\beta c + 3\alpha\gamma)\alpha_1 &= 0,\\
-2\alpha^3\beta\lambda\alpha_1 + \beta\alpha^2\lambda\alpha_1^2 &= 0,\\
-2\alpha^3\beta\alpha_1 + \beta\alpha^2\alpha_1^2 &= 0.
\end{aligned}$$

Solving the above system with the aid of Maple 10, we have the following set of solutions:

$$\alpha_1 = 2\alpha, \;\; c_1 = 0, \;\; c = -\frac{\alpha}{2\beta}(4\alpha^2\beta - \alpha^2\beta\lambda^2 + 3\gamma). \tag{28}$$

Substituting (28) together with the solution of Eq. (4) into (27), we obtain three types of traveling wave solutions of Eq. (24) as follows:
when $\lambda^2 - 4\mu > 0$,

$$u(\xi) = \alpha\sqrt{\lambda^2 - 4\mu}\left(\frac{A_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + A_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{A_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + A_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right) - \alpha\lambda + \alpha_0, \tag{29}$$

where A_1 and A_2 are arbitrary constants.
when $\lambda^2 - 4\mu < 0$,

$$u(\xi) = \alpha\sqrt{4\mu - \lambda^2}\left(\frac{A_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi - A_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{A_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + A_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right) - \alpha\lambda + \alpha_0, \tag{30}$$

when $\lambda^2 - 4\mu = 0$,

$$u(\xi) = 2\alpha\left(\frac{A_2}{A_1 + A_2\xi}\right) - \alpha\lambda + \alpha_0. \tag{31}$$

In Particular, if we take $A_1 = 0$, $A_2 \neq 0$, $\lambda = 2k$ and $\mu = 0$ then the solution (29) becomes

$$u(\xi) = 2k\alpha[\tanh k\xi - 1] + \alpha_0, \tag{32}$$

which is the exact solution of Jimbo-Miwa equation same as obtained in [38].

4 Conclusions

In this work, we have successfully obtained the three types of traveling wave solutions in terms of hyperbolic, trigonometric and rational functions for the (2+1)-dimensional NNVE and (3+1)-dimensional Jimbo-Miwa equations by using the $(\frac{G'}{G})$-expansion method. The solution of these nonlinear evolution equations have many potential applications in Mathematical Physics. It is also shown that the performance of this method is effective, simple and direct to obtain the exact solutions of the nonlinear evolution equations with the help of computer algebra system. Compared with the others methods used in [1–11], one can see that the $(\frac{G'}{G})$-expansion method is not only simple and straightforward, but also avoid tedious algebraic calculations.

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