ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.8(2009) No.4,pp.435-447

The Traveling Wave Solutions for Nonlinear Partial Differential Equations Using the $(\frac{G'}{G})$-expansion Method

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(Received 10 June 2009, accepted 13 November 2009)

Abstract: In the present paper, we construct the traveling wave solutions involving parameters for some nonlinear evolution equations in mathematical physics via the (1+1)-dimensional nonlinear dispersive equation, the (1+1)-dimensional nonlinear Benjamin-Bona Mahony equation, the (1+1)-dimensional nonlinear KdV coupled equations and the (1+1)-dimensional nonlinear Ostrovsky equation by using a new approach, namely the $(\frac{G'}{G})$-expansion method, where $G = G(\xi)$ satisfies a second order linear ordinary differential equation. When the parameters are taken special values, the solitary waves are derived from the traveling waves. The traveling wave solutions are expressed by hyperbolic, trigonometric and rational functions.

Keywords: the $(\frac{G'}{G})$-expansion method; traveling wave solutions; dispersive equation; Benjamin-Bona Mahony equation; KdV coupled equations; Ostrovsky equation; solitary wave solutions

AMS: subject classification: 35K99; 35P05; 35P99.

1 Introduction

In recent years, the exact solutions of nonlinear PDEs have been investigated by many authors (see for example [1-52]) who are interested in nonlinear physical phenomena. Many powerful methods have been presented such as the tanh-function method [1, 10, 19, 34, 36, 44, 45, 50], the F-function expansion method [2, 27, 28, 47], the inverse scattering transform [3], the direct algebraic method [9], the exp-function expansion method [5, 11, 35, 48, 49], the Jacobi elliptic function expansion [8, 17, 33, 38, 43], the Backlund transform [7, 20, 22, 23], the homogeneous balance method [21], the Riccati equation [14, 31, 32], the extended sinh-cosh and sine-cosine methods [26], the sub-ODE method [18, 29], the complex hyperbolic function method [41], the truncated Painlevé expansion [13, 46], the homotopy perturbation method [42] and others. In the present paper, we shall use a new method which is called the $(\frac{G'}{G})$-expansion method [4, 30, 37, 39, 40, 51, 52]. This method has been proposed by the Chinese mathematicians Wang et al [30] for which the traveling wave solutions of the nonlinear evolution equations are obtained. The main idea of this method is that the traveling wave solutions of the nonlinear evolution equations can be expressed by polynomials in $(\frac{G'}{G})$, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$ with λ , μ and V being constants while $' = \frac{d}{d\xi}$. The degree of these polynomials can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equations. The coefficients of these polynomials can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method.

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2 Description of the $\left(\frac{G'}{G}\right)$-expansion method

Suppose we have the following nonlinear partial differential equation:

$$P(u,\ u_t,\ u_x,\ u_{tt},\ u_{xt},\ u_{xx},\dots)=0, \tag{1}$$

where $u=u(x,t)$ is unknown function, P is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. Now, we give the main steps proposed in [30] for solving Eq.(1) using the $(\frac{G'}{G})$-expansion method:
Step 1. The traveling wave variable

$$u(x,t)=u(\xi), \qquad \xi=x-Vt, \tag{2}$$

permits us reducing Eq.(1) to an ODE for $u=u(\xi)$ in the form

$$F(u,-Vu',u',V^2u'',-V^2u'',u'',\dots)=0, \tag{3}$$

where V is a constant while F is a polynomial in $u(\xi)$ and its total derivatives.
Step 2. Suppose the solution of Eq.(3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi)=\sum_{i=0}^{n}\alpha_i\left(\frac{G'}{G}\right)^i, \tag{4}$$

where $G=G(\xi)$ satisfies the following second order linear ODE :

$$G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0, \tag{5}$$

while $\alpha_i\ (i=0,1,\dots,n),\ \lambda$ and μ are constants to be determined provided $\alpha_n\neq 0$. The positive integer "n" can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq.(3).
Step 3. Substituting (4) into Eq.(3) and using Eq.(5), collecting all terms with the same power of $(\frac{G'}{G})$ together and then equating each coefficient of the resulted polynomial to zero, yield a set of algebraic equations for α_i , V, λ and μ.
Step 4. Since the general solution of Eq.(5) has been well known for us, then substituting α_i , V and the general solution of Eq.(5) into (4), we have the traveling wave solutions of the nonlinear partial differential equation (1).

3 Some applications

In this section, we apply the $(\frac{G'}{G})$-expansion method to construct the traveling wave solutions for the dispersive equation, the Benjamin-Bona Mahony equation, the KdV coupled equations and the Ostrovsky equation which are very important nonlinear evolution equations in mathematical physics and have been paid attention by many researchers.

3.1 Example 1: The (1+1)-dimensional nonlinear dispersive equation

We start with the following (1+1)-dimensional nonlinear dispersive equation [11] in the form:

$$u_t-\beta u^2u_x+u_{xxx}=0\ , \tag{6}$$

where β is a nonzero positive constant. This equation is called the modified KdV equation, which arises in the process of understanding the role of nonlinear dispersion and in the formation of structures like liquid drops , and it exhibits compaction: solitons with compact support. He et al [11] have used the Exp-function method to find the traveling wave solutions of Eq.(6). Let us now solve Eq.(6) by the proposed method. To this end, we see that the traveling wave variable (2) permits us converting Eq.(6) into the following ODE:

$$-Vu'-\beta u^2\,u'+u'''=0. \tag{7}$$

By integrating Eq.(7) with respect to ξ once, yields

$$C - Vu - \frac{\beta}{3}u^3 + u'' = 0, \tag{8}$$

where C is an integration constant. Suppose the solution of the ODE (8) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{n} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{9}$$

where $\alpha_i\ (i = 0, 1, \ldots, n)$ are arbitrary constants provided $\alpha_n \neq 0$, while $G(\xi)$ satisfies the second order linear ODE (5). Considering the homogeneous balance between highest order derivatives and nonlinear terms in Eq.(8), we get n=1. Thus, we have

$$u(\xi) = \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0. \tag{10}$$

Consequently, we have

$$u^3 = {\alpha_1}^3 \left(\frac{G'}{G}\right)^3 + 3\alpha_0\alpha_1^2 \left(\frac{G'}{G}\right)^2 + 3\alpha_0^2\alpha_1 \left(\frac{G'}{G}\right) + {\alpha_0}^3, \tag{11}$$

$$u'' = 2\alpha_1 \left(\frac{G'}{G}\right)^3 + 3\lambda\alpha_1 \left(\frac{G'}{G}\right)^2 + (\alpha_1\lambda^2 + 2\mu\alpha_1) \left(\frac{G'}{G}\right) + \alpha_1\lambda\mu\,. \tag{12}$$

Substituting (10)-(12) into Eq.(8), collecting all terms with the same powers of $(\frac{G'}{G})$ and setting them to zero, we have the following system of algebraic equations:

$$\begin{aligned}
0 &: \; C - V\alpha_0 - \frac{\beta}{3}\alpha_0^3 + \lambda\mu\alpha_1 = 0, \\
1 &: \; -V\alpha_1 - \beta\alpha_1\alpha_0^2 + \lambda^2\alpha_1 + 2\mu\alpha_1 = 0, \\
2 &: \; -\beta\alpha_0\alpha_1^2 + 3\lambda\alpha_1 = 0, \\
3 &: \; -\frac{\beta}{3}\alpha_1^3 + 2\alpha_1 = 0.
\end{aligned}$$

Solving these algebraic equations yields

$$\alpha_1 = \pm\sqrt{\frac{6}{\beta}} \quad , \qquad \alpha_0 = \pm\,\lambda\sqrt{\frac{3}{2\beta}} \quad , \qquad V = \frac{1}{2}(4\mu - \lambda^2) \quad , \qquad C = 0\,. \tag{13}$$

Substituting (13) into (10) we obtain

$$u(\xi) = \pm\sqrt{\frac{6}{\beta}} \left(\frac{G'}{G}\right) \pm\; \lambda\sqrt{\frac{3}{2\beta}}\,, \tag{14}$$

where

$$\xi = x - \frac{1}{2}(4\mu - \lambda^2)\, t. \tag{15}$$

Solving Eq.(5), we deduce for $\lambda^2 - 4\mu > 0$ that

$$\frac{G'}{G} = \frac{1}{2}\sqrt{\lambda^2 - 4\mu} \left(\frac{A\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\,\right) + B\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\,\right)}{A\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\,\right) + B\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\,\right)}\right) - \frac{\lambda}{2}, \tag{16}$$

where A and B are arbitrary constants. From (10), (13) and the general solutions of Eq.(5), we deduce the following traveling wave solutions:
Case 1. If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi) = \pm\sqrt{\frac{3(\lambda^2 - 4\mu)}{2\beta}} \left(\frac{A\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\,\right) + B\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\,\right)}{A\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\,\right) + B\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\,\right)}\right). \tag{17}$$

Case 2. If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi) \pm \sqrt{\frac{3(4\mu-\lambda^2)}{2\beta}} \left(\frac{B\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) - A\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{A\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + B\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)} \right). \tag{18}$$

Case 3. If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = \pm \sqrt{\frac{6}{\beta}} \left(\frac{B}{A + B\xi} \right). \tag{19}$$

In particular, if $A = 0$, $B \neq 0$, $\lambda > 0$ and $\mu = 0$, then we deduce from (17) that:

$$u(\xi) = \pm \sqrt{\frac{3}{2\beta}}\, \lambda\, \coth\left(\frac{\lambda}{2}\xi\right), \tag{20}$$

while, if $A \neq 0$, $A^2 > B^2$, $\lambda > 0$ and $\mu = 0$, then we deduce that:

$$u(\xi) = \pm \sqrt{\frac{3}{2\beta}}\, \lambda\, \tanh\left(\frac{\lambda}{2}\xi + \xi_0\right), \tag{21}$$

where $\xi_0 = \tanh^{-1}(\frac{B}{A})$. The solutions (20) and (21) represent the solitary wave solutions of the (1+1)-dimensional nonlinear dispersive equation (6).

3.2 Example 2: The (1+1)-dimensional nonlinear Benjamin-Bona Mahony equation

In this subsection, we study the following (1+1)-dimensional nonlinear Benjamin-Bona Mahony equation [35] in the form :

$$u_t + u_x - \alpha\, u^2 u_x + u_{xxx} = 0, \tag{22}$$

where α is a nonzero positive constant. This equation was first derived to describe an approximation for surface long waves in nonlinear media. It can also characterize the hydromagnetic waves in cold plasma, a caustic waves in inharmonic crystals and acoustic-gravity waves in compressible fluids. Yusufoglu [35] has used the Exp-function method to find the traveling wave solutions of Eq.(22). Let us now solve Eq.(22) by the proposed method. To this end, we see that the traveling wave variable (2) permits us converting Eq.(22) into the following ODE:

$$(1-V)\, u' - \alpha\, u^2\, u' + u''' = 0. \tag{23}$$

Integrating Eq.(23) with respect to ξ once, we get

$$C + (1-V)\, u - \frac{1}{3}\alpha\, u^3 + u'' = 0, \tag{24}$$

where C is an integration constant. Considering the homogeneous balance between highest order derivatives and nonlinear terms in Eq.(24), we deduce that the solution $u(\xi)$ of Eq.(24) has the same form of (10). Substituting (10)-(12) into Eq.(24), collecting all terms with the same powers of $(\frac{G'}{G})$ and setting them to zero, we have the following system of algebraic equations:

$$\begin{aligned}
0 &: \quad C - V\alpha_0 + \alpha_0 - \frac{1}{3}\alpha\alpha_0^3 + \lambda\mu\alpha_1 = 0,\\
1 &: \quad \alpha_1 - V\alpha_1 - \alpha\alpha_1\alpha_0^2 + \lambda^2\alpha_1 + 2\mu\alpha_1 = 0,\\
2 &: \quad -\alpha\alpha_0\alpha_1^2 + 3\lambda\alpha_1 = 0,\\
3 &: \quad -\frac{1}{3}\alpha\alpha_1^3 + 2\alpha_1 = 0.
\end{aligned}$$

Solving these algebraic equations yields

$$\alpha_1 = \pm\sqrt{\frac{6}{\alpha}} \quad , \quad \alpha_0 = \pm\,\lambda\sqrt{\frac{3}{2\alpha}} \quad , \quad V = \frac{1}{2}\left(2 - \lambda^2 + 4\mu\right) \quad , \quad C = 0 \quad . \tag{25}$$

From (10), (25) and the general solutions of Eq.(5), we deduce the following traveling wave solutions of Eq.(22):
Case 1. If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi) = \pm\sqrt{\frac{3\left(\lambda^2 - 4\mu\right)}{2\alpha}}\left(\frac{A\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\right) + B\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\right)}{A\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\right) + B\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2 - 4\mu}\right)}\right). \tag{26}$$

Case 2. If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi) = \pm\sqrt{\frac{3\left(4\mu - \lambda^2\right)}{2\alpha}}\left(\frac{B\cos\left(\frac{\xi}{2}\sqrt{4\mu - \lambda^2}\right) - A\sin\left(\frac{\xi}{2}\sqrt{4\mu - \lambda^2}\right)}{A\cos\left(\frac{\xi}{2}\sqrt{4\mu - \lambda^2}\right) + B\sin\left(\frac{\xi}{2}\sqrt{4\mu - \lambda^2}\right)}\right). \tag{27}$$

Case 3. If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = \pm\sqrt{\frac{3}{2\alpha}}\left(\frac{B}{A + B\xi}\right). \tag{28}$$

In particular, if $A = 0$, $B \neq 0$, $\lambda > 0$ and $\mu = 0$, then we deduce from (26) that:

$$u(\xi) = \pm\sqrt{\frac{3}{2\alpha}}\,\lambda\,\coth\left(\frac{\lambda}{2}\xi\right), \tag{29}$$

while, if $A \neq 0$, $A^2 > B^2$, $\lambda > 0$ and $\mu = 0$, then we deduce that:

$$u(\xi) = \pm\sqrt{\frac{3}{2\alpha}}\,\lambda\,\tanh\left(\frac{\lambda}{2}\xi + \xi_0\right), \tag{30}$$

where $\xi_0 = \tanh^{-1}(\frac{B}{A})$. The solutions (29) and (30) represent the solitary wave solutions of the (1+1)-dimensional nonlinear Benjamin-Bona Mahony equation (22).

3.3 Example 3: The (1+1)-dimensional nonlinear KdV coupled equations

In this subsection, we study the following (1+1)-dimensional nonlinear KdV coupled [12] equations in the form:

$$u_t - \alpha u_{xxx} - 6\alpha u u_x + 2\beta v v_x = 0, \tag{31}$$
$$v_t + v_{xxx} + 3u v_x = 0, \tag{32}$$

where α and β are nonzero positive constants. The coupled KdV equations (31) and (32) are introduced by Hirota-Satsuma [12] and describe interactions of two long waves with different dispersion relations. In [12], the authors showed that for all values of α and β, the system (31) and (32) possesses three conservation lows and a solitary wave solution. In [15], the authors have found the exact and numerical traveling wave solutions of this system using the decomposition method. Let us now solve the system (31) and (32) by the proposed method. To this end, we see that the traveling wave variable (2) permits us converting Eqs.(31) and (32) into the following ODEs:

$$Vu' + \alpha\, u''' + 6\alpha u u' + 2\beta v v' = 0, \tag{33}$$
$$-Vv' + v''' + 3u v' = 0. \tag{34}$$

Integrating Eq.(33) with respect to ξ once, yields

$$C + Vu + \alpha\, u'' + 3\alpha u^2 + \beta v^2 = 0, \tag{35}$$

where C is an integration constant. Considering the homogeneous balance between highest order derivatives and nonlinear terms in Eqs.(34) and (35), we have

$$\begin{aligned} u(\xi) &= \alpha_2\left(\frac{G'}{G}\right)^2+\alpha_1\left(\frac{G'}{G}\right)+\alpha_0, \qquad \alpha_2\neq 0, && (36)\\ v(\xi) &= \beta_2\left(\frac{G'}{G}\right)^2+\beta_1\left(\frac{G'}{G}\right)+\beta_0, \qquad \beta_2\neq 0. && (37)\end{aligned}$$

Consequently, we find that

$$\begin{aligned}
u^2 &= \alpha_2^2\left(\frac{G'}{G}\right)^4+2\alpha_1\alpha_2\left(\frac{G'}{G}\right)^3+(\alpha_1^2+2\alpha_0\alpha_2)\left(\frac{G'}{G}\right)^2+2\alpha_0\alpha_1\left(\frac{G'}{G}\right)+\alpha_0^2, && (38)\\
v^2 &= \beta_2^2\left(\frac{G'}{G}\right)^4+2\beta_1\beta_2\left(\frac{G'}{G}\right)^3+(\beta_1^2+2\beta_0\beta_2)\left(\frac{G'}{G}\right)^2+2\beta_0\beta_1\left(\frac{G'}{G}\right)+\beta_0^2, && (39)\\
v' &= -2\beta_2\left(\frac{G'}{G}\right)^3-(2\lambda\beta_2+\beta_1)\left(\frac{G'}{G}\right)^2-(2\mu\beta_2+\lambda\beta_1)\left(\frac{G'}{G}\right)-\mu\beta_1\ , && (40)\\
u'' &= 6\alpha_2\left(\frac{G'}{G}\right)^4+(10\ \lambda\ \alpha_2+2\alpha_1)\left(\frac{G'}{G}\right)^3+(8\mu\alpha_2+4\ \lambda^2\ \alpha_2+3\ \lambda\ \alpha_1)\left(\frac{G'}{G}\right)^2\\
&+ (6\ \lambda\ \mu\ \alpha_2+2\ \mu\ \alpha_1+\lambda^2\ \alpha_1)\left(\frac{G'}{G}\right)+2\ \mu^2\ \alpha_2+\lambda\ \mu\ \alpha_1\ , && (41)\\
v''' &= -24\beta_2\left(\frac{G'}{G}\right)^5-(54\lambda\beta_2+6\beta_1)\left(\frac{G'}{G}\right)^4-\left(40\mu\beta_2+38\lambda^2\beta_2+12\lambda\beta_1\right)\left(\frac{G'}{G}\right)^3\\
&- \left(52\lambda\mu\beta_2+8\mu\beta_1+8\lambda^3\beta_2+7\lambda^2\beta_1\right)\left(\frac{G'}{G}\right)^2-(\lambda^3\beta_1+14\lambda^2\mu\beta_2+8\lambda\mu\beta_1\\
&+ 16\mu^2\beta_2)\left(\frac{G'}{G}\right)-\left(6\lambda\mu^2\beta_2+2\mu^2\beta_1+\lambda^2\mu\beta_1\right)\ , && (42)\\
uv' &= -2\alpha_2\beta_2\left(\frac{G'}{G}\right)^5-(2\alpha_1\beta_2+2\lambda\alpha_2\beta_2+\alpha_2\beta_1)\left(\frac{G'}{G}\right)^4-(2\alpha_0\beta_2+2\lambda\alpha_1\beta_2\\
&+ \alpha_1\beta_1+2\mu\alpha_2\beta_2+\lambda\beta_1\alpha_2)\left(\frac{G'}{G}\right)^3-(2\lambda\alpha_0\beta_2+\alpha_0\beta_1+2\mu\beta_2\alpha_1+\lambda\alpha_1\beta_1\\
&+ \mu\alpha_2\beta_1)\left(\frac{G'}{G}\right)^2-(2\mu\alpha_0\beta_2+\lambda\alpha_0\beta_1+\mu\alpha_1\beta_1)\left(\frac{G'}{G}\right)-\mu\alpha_0\beta_1\ . && (43)
\end{aligned}$$

Substituting (36) and (38)–(43) into Eqs.(34) and (35), collecting all terms with the same power of $(\frac{G'}{G})$ and setting them to zero, we have the following system of algebraic equations:

$$\begin{aligned}
0 &: \ V\mu\beta_1-6\lambda\mu^2\beta_2-2\mu^2\beta_1-\mu\lambda^2\beta_1-3\mu\alpha_0\beta_1=0,\\
1 &: \ \lambda V\beta_1+2\mu V\beta_2-14\mu\lambda^2\beta_2-8\lambda\mu\beta_1-\lambda^3\beta_1-16\mu^2\beta_2-3\mu\alpha_1\beta_1-3\lambda\alpha_0\beta_1-6\mu\alpha_0\beta_2=0,\\
2 &: \ 2\lambda V\beta_2+V\beta_1-8\lambda^3\beta_2-52\lambda\mu\beta_2-8\mu\beta_1-7\lambda^2\beta_1-3\mu\alpha_2\beta_1-3\lambda\alpha_1\beta_1-6\mu\alpha_1\beta_2\\
&\quad -6\lambda\alpha_0\beta_2-3\alpha_0\beta_1=0,\\
3 &: \ 2V\beta_2-38\lambda^2\beta_2-12\lambda\beta_1-40\mu\beta_2-6\alpha_0\beta_2-6\lambda\alpha_1\beta_2-3\lambda\alpha_2\beta_1-6\mu\alpha_2\beta_2-3\alpha_1\beta_1=0,\\
4 &: \ -54\lambda\beta_2-6\beta_1-6\lambda\alpha_2\beta_2-6\alpha_1\beta_2-3\alpha_2\beta_1=0,\\
5 &: \ 24\beta_2+6\alpha_2\beta_2=0,\\
0 &: \ C+V\alpha_0+2\mu^2\alpha\alpha_2+\lambda\mu\alpha\alpha_1+3\alpha\alpha_0^2-\beta\beta_0^2=0,\\
1 &: \ V\alpha_1+6\lambda\mu\alpha\alpha_2+2\mu\alpha\alpha_1+\lambda^2\alpha\alpha_1+6\alpha\alpha_0\alpha_1-2\beta\beta_0\beta_1=0,\\
2 &: \ V\alpha_2+8\mu\alpha\alpha_2+4\lambda^2\alpha\alpha_2+3\lambda\alpha\alpha_1+3\alpha\alpha_1^2+6\alpha\alpha_0\alpha_2-\beta\beta_1^2-2\beta\beta_0\beta_2=0,\\
3 &: \ 10\lambda\alpha\alpha_2+2\alpha\alpha_1+6\alpha\alpha_1\alpha_2-2\beta\beta_1\beta_2=0,\\
4 &: \ 6\alpha\alpha_2+3\alpha\alpha_2^2-\beta\beta_2^2=0,
\end{aligned}$$

Solving the above algebraic equations by Maple, yields

$$\begin{aligned}
\alpha_2 &= -4\,,\ \ \alpha_1 = -4\lambda\,,\ \ V = 3\alpha_0 + \lambda^2 + 8\mu \quad, \beta_2 = \pm\, 2\sqrt{\frac{6\alpha}{\beta}}\,,\ \ \beta_1 = \pm\, 2\lambda\sqrt{\frac{6\alpha}{\beta}}\,,\\
\beta_0 &= \mp\,\frac{1}{\sqrt{6\alpha\beta}}\left[3\alpha_0\,(2\alpha+1) + (\lambda^2+8\mu)\,(\alpha+1)\right]\,,\ C = \frac{1}{6\alpha}\Bigg\{16\mu^2\left[\alpha\,(7\alpha+8)+4\right]\\
&+\ \lambda^4\,(\alpha+1)^2 + 3\alpha_0\left[2\alpha\,(\alpha+1)+1\right]\left[2\left(\lambda^2+8\mu\right)+3\alpha_0\right]8\lambda^2\mu\left[\alpha\,(5\alpha+4)+2\right]\Bigg\}\,. \qquad (44)
\end{aligned}$$

From (36)–(37), (44) and the general solutions of Eq.(5), we deduce the following traveling wave solutions:
Case 1. If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi) = -(\lambda^2-4\mu)\left(\frac{A\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + B\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{A\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + B\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right)^2 + \lambda^2 + \alpha_0, \qquad (45)$$

$$\begin{aligned}
v(\xi) &= \pm\,\frac{1}{2}\,(\lambda^2-4\mu)\,\sqrt{\frac{6\alpha}{\beta}}\left(\frac{A\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + B\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{A\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + B\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right)^2\\
&\mp\ \frac{1}{2}\,\lambda^2\,\sqrt{\frac{6\alpha}{\beta}}\ \mp\ \frac{1}{\sqrt{6\alpha\beta}}\left[3\alpha_0\,(2\alpha+1) + (\lambda^2+8\mu)\,(\alpha+1)\right]\,. \qquad (46)
\end{aligned}$$

Case 2. If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi) = -(4\mu-\lambda^2)\left(\frac{B\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) - A\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{A\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + B\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}\right)^2 + \lambda^2 + \alpha_0, \qquad (47)$$

$$\begin{aligned}
v(\xi) &= \pm\,\frac{1}{2}\,(4\mu-\lambda^2)\,\sqrt{\frac{6\alpha}{\beta}}\left(\frac{B\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) - A\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{A\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + B\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}\right)^2\\
&\mp\ \frac{1}{2}\,\lambda^2\,\sqrt{\frac{6\alpha}{\beta}}\ \mp\ \frac{1}{\sqrt{6\alpha\beta}}\left[3\alpha_0\,(2\alpha+1) + (\lambda^2+8\mu)\,(\alpha+1)\right]\,. \qquad (48)
\end{aligned}$$

Case 3. If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = -4\left(\frac{B}{A+B\xi}\right)^2 + \lambda^2 + \alpha_0\,, \qquad (49)$$

$$v(\xi) = \pm\, 2\sqrt{\frac{6\alpha}{\beta}}\left(\frac{B}{A+B\xi}\right)^2 \mp \frac{1}{2}\lambda^2\,\sqrt{\frac{6\alpha}{\beta}} \mp \frac{1}{\sqrt{6\alpha\beta}}\left[3\alpha_0\,(2\alpha+1) + (\lambda^2+8\mu)\,(\alpha+1)\right]\,, \qquad (50)$$

where

$$\xi = x - \left(3\alpha_0 + \lambda^2 + 8\mu\right)t. \qquad (51)$$

In particular if $A = 0$, $B \neq 0$, $\lambda > 0$ and $\mu = 0$, then we deduce from (45) and (46) that:

$$\begin{aligned}
u(\xi) &= -\lambda^2\mathrm{csch}^2\left(\frac{\lambda}{2}\xi\right) + \alpha_0\,, \qquad (52)\\
v(\xi) &= \pm\,\frac{\lambda^2}{2}\sqrt{\frac{6\alpha}{\beta}}\,\mathrm{csch}^2\left(\frac{\lambda}{2}\xi\right) \mp\ \frac{1}{\sqrt{6\alpha\beta}}\left[3\alpha_0\,(2\alpha+1) + \lambda^2\,(\alpha+1)\right]\,, \qquad (53)
\end{aligned}$$

while, if $A \neq 0$, $\lambda > 0$ and $\mu = 0$, then we deduce that:

$$\begin{aligned}
u(\xi) &= \lambda^2\mathrm{sech}^2\left(\frac{\lambda}{2}\,\xi + \xi_0\right) + \alpha_0\,, \qquad (54)\\
v(\xi) &= \mp\,\frac{1}{2}\,\lambda^2\sqrt{\frac{6\alpha}{\beta}}\,\mathrm{sech}^2\left(\frac{\lambda}{2}\,\xi + \xi_0\right) \mp\ \frac{1}{\sqrt{6\alpha\beta}}\left[3\alpha_0\,(2\alpha+1) + \lambda^2\,(\alpha+1)\right]\,, \qquad (55)
\end{aligned}$$

where $\xi_0 = \tanh^{-1}(\frac{B}{A})$. The solutions (52)-(55) represent the solitary wave solutions of the (1+1)-dimensional nonlinear KdV coupled equations (31)-(32).

3.4 Example 4: The (1+1)-dimensional nonlinear Ostrovsky equation

In this subsection, we study the following (1+1)-dimensional nonlinear Ostrovsky equation [6, 16, 24, 25] in the form:

$$uu_{xxt} - u_x u_{xt} + u^2 u_t = 0. \tag{56}$$

This equation has been discovered by Vakhnenko and Parkers [24]. In [25] this equation has been found completely integrable by inverse scattering method. Its particular solutions can be also constructed by means of Hirota and Backlund transformations. Bekir and Yusufoglu [6] have used hyperbolic tangent method and Exp-function method to find the traveling wave solutions for Eq.(56). Köroğlu and Öziş [16] have reduced a novel traveling wave solutions for this equation by using Exp-function method. This equation is a model for weakly nonlinear surface and internal waves in a rotating ocean. Let us now solve Eq.(56) by the proposed method. To this end, we use the traveling wave variable (2) to convert Eq.(56) into the following ODE:

$$-(uu'')' + 2u'u'' - u^2 u' = 0. \tag{57}$$

Integrating Eq.(57) with respect to ξ, one has

$$C + 3uu'' - 3(u')^2 + u^3 = 0. \tag{58}$$

Considering the homogeneous balance between highest order derivatives and nonlinear terms in Eq.(58), we have the solution $u(\xi)$ has the same form of (36). Consequently, we have:

$$\begin{aligned} u^3 &= {\alpha_2}^3 \left(\frac{G'}{G}\right)^6 + 3\alpha_1{\alpha_2}^2 \left(\frac{G'}{G}\right)^5 + 3({\alpha_1}^2\alpha_2 + \alpha_0{\alpha_2}^2) \left(\frac{G'}{G}\right)^4 + (6\alpha_0\alpha_1\alpha_2 \\ &+ \alpha_1^3) \left(\frac{G'}{G}\right)^3 + 3(\alpha_2\alpha_0^2 + \alpha_0\alpha_1^2) \left(\frac{G'}{G}\right)^2 + 3\alpha_1\alpha_0^2 \left(\frac{G'}{G}\right) + {\alpha_0}^3, \end{aligned} \tag{59}$$

$$\begin{aligned} (u')^2 &= 4{\alpha_2}^2 \left(\frac{G'}{G}\right)^6 + 4(\alpha_1\alpha_2 + 2\lambda{\alpha_2}^2) \left(\frac{G'}{G}\right)^5 + [4(\lambda^2 + 2\mu){\alpha_2}^2 + 8\lambda\alpha_1\alpha_2 \\ &+ {\alpha_1}^2] \left(\frac{G'}{G}\right)^4 + 2(4\lambda\mu{\alpha_2}^2 + 4\mu\alpha_1\alpha_2 + 2\lambda^2\alpha_1\alpha_2 + \lambda{\alpha_1}^2) \left(\frac{G'}{G}\right)^3 + (\lambda^2{\alpha_1}^2 \\ &+ 8\lambda\mu\alpha_1\alpha_2 + 2\mu{\alpha_1}^2 + 4\mu^2{\alpha_2}^2) \left(\frac{G'}{G}\right)^2 + 2(\lambda\mu{\alpha_1}^2 + 2\mu^2\alpha_1\alpha_2) \left(\frac{G'}{G}\right) + \mu^2{\alpha_1}^2, \end{aligned} \tag{60}$$

$$\begin{aligned} uu'' &= 6{\alpha_2}^2 \left(\frac{G'}{G}\right)^6 + 2(4\alpha_1\alpha_2 + 5\lambda{\alpha_2}^2) \left(\frac{G'}{G}\right)^5 + [4(\lambda^2 + 2\mu){\alpha_2}^2 + 13\lambda\alpha_1\alpha_2 \\ &+ 2{\alpha_1}^2 + 6\alpha_0\alpha_2] \left(\frac{G'}{G}\right)^4 + [5(2\lambda\alpha_0 + 2\mu\alpha_1 + \lambda^2\alpha_1)\alpha_2 + 2\alpha_0\alpha_1 + 3\lambda{\alpha_1}^2 \\ &+ 6\lambda\mu{\alpha_2}^2] \left(\frac{G'}{G}\right)^3 + [(4\lambda^2\alpha_0 + 8\mu\alpha_0 + 7\lambda\mu\alpha_1)\alpha_2 + (\lambda^2 + 2\mu){\alpha_1}^2 + 3\lambda\alpha_0\alpha_1 \\ &+ 2\mu^2{\alpha_2}^2] \left(\frac{G'}{G}\right)^2 + [(\lambda^2 + 2\mu)\alpha_0\alpha_1 + 6\lambda\mu\alpha_0\alpha_2 + 2\mu^2\alpha_1\alpha_2 + \lambda\mu{\alpha_1}^2] \left(\frac{G'}{G}\right) \\ &+ \mu\,\alpha_0(\lambda\alpha_1 + 2\mu\alpha_2). \end{aligned} \tag{61}$$

Substituting (59)-(61) into Eq.(58), collecting all terms with the same power of $(\frac{G'}{G})$ and setting them to zero, we have the following system of algebraic equations:

$$\begin{aligned}
0 &: C + 3\lambda\mu\alpha_0\alpha_1 - 3\mu^2{\alpha_1}^2 + {\alpha_0}^3 + 6\mu^2\alpha_0\alpha_2 = 0,\\
1 &: 3\lambda^2\alpha_0\alpha_1 + 6\mu\alpha_0\alpha_1 + 3{\alpha_0}^2\alpha_1 - 3\lambda\mu{\alpha_1}^2 + 18\lambda\mu\alpha_0\alpha_1 - 6\mu^2\alpha_1\alpha_2 = 0,\\
2 &: 9\lambda\alpha_0\alpha_1 + 3\alpha_0{\alpha_1}^2 + 12\lambda^2\alpha_0\alpha_2 + 24\mu\alpha_0\alpha_2 + 3{\alpha_0}^2\alpha_2 - 3\lambda\mu\alpha_1\alpha_2 - 6\mu^2{\alpha_2}^2 = 0,\\
3 &: 6\alpha_0\alpha_1 + 3\lambda{\alpha_1}^2 + {\alpha_1}^3 + 30\lambda\alpha_0\alpha_2 + 3\lambda^2\alpha_1\alpha_2 + 6\mu\alpha_1\alpha_2 + 6\alpha_0\alpha_1\alpha_2 - 6\lambda\mu{\alpha_2}^2 = 0,\\
4 &: 3{\alpha_1}^2 + 18\alpha_0\alpha_2 + 15\lambda\alpha_1\alpha_2 + 3{\alpha_1}^2\alpha_2 + 3\alpha_0{\alpha_2}^2 = 0,\\
5 &: 12\alpha_1\alpha_2 + 6\lambda{\alpha_2}^2 + 3\alpha_1{\alpha_2}^2 = 0,\\
6 &: 6\alpha_2^2 + \alpha_2^3 = 0.
\end{aligned}$$

Solving the above algebraic equations yields

$$\alpha_2 = -\,6,\ \alpha_1 = -6\,\lambda\,,\ \alpha_0 = -6\,\mu\,,\quad C = 0\,, \tag{62}$$

or

$$\alpha_2 = -\,6,\ \alpha_1 = -6\,\lambda\,,\ \alpha_0 = -\left(\lambda^2 + 2\mu\right)\,,\ C = (\lambda^2 - 4\mu)^3\,. \tag{63}$$

From (36), (61)-(62) and the general solutions of Eq.(5), we deduce the following traveling wave solutions:
Case 1. If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi) = -\,\frac{3(\lambda^2-4\mu)}{2}\left(\frac{A\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + B\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{A\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + B\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right)^2 + \frac{3}{2}\lambda^2 - 6\mu\,, \tag{64}$$

or

$$u(\xi) = -\,\frac{3(\lambda^2-4\mu)}{2}\left(\frac{A\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + B\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}{A\cosh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right) + B\sinh\left(\frac{\xi}{2}\sqrt{\lambda^2-4\mu}\right)}\right)^2 + \frac{1}{2}\lambda^2 - 2\mu\,. \tag{65}$$

Case 2. If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi) = -\,\frac{3(\lambda^2-4\mu)}{2}\left(\frac{B\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) - A\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{A\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + B\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}\right)^2 + \frac{3}{2}\lambda^2 - 6\mu\,, \tag{66}$$

or

$$u(\xi) = -\,\frac{3(\lambda^2-4\mu)}{2}\left(\frac{B\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) - A\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}{A\cos\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right) + B\sin\left(\frac{\xi}{2}\sqrt{4\mu-\lambda^2}\right)}\right)^2 + \frac{1}{2}\lambda^2 - 2\mu\,. \tag{67}$$

Case 3. If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi) = -6\,\left(\frac{B}{A+B\xi}\right)^2 + \frac{3}{2}\lambda^2 - 6\mu, \tag{68}$$

or

$$u(\xi) = -6\,\left(\frac{B}{A+B\xi}\right)^2 + \frac{1}{2}\lambda^2 - 2\mu, \tag{69}$$

where

$$\xi = x - V\,t. \tag{70}$$

In particular , if $A = 0, B \neq 0, \lambda > 0$ and $\mu = 0$, then we deduce from (64) and (65) that

$$u(\xi) = -\frac{3}{2}\lambda^2\mathrm{csch}^2\left(\frac{\lambda}{2}\,\xi\right)\,, \tag{71}$$

or

$$u(\xi) = -\frac{3}{2}\lambda^2 \text{csch}^2\left(\frac{\lambda}{2}\lambda\, \xi\right) - \lambda^2 \,, \tag{72}$$

while, if $A \neq 0$, $\lambda > 0$ and $\mu = 0$, then we deduce that:

$$u(\xi) = \frac{3}{2}\lambda^2\, \text{sech}^2\left(\frac{\lambda}{2}\,\xi + \xi_0\right) \,, \tag{73}$$

or

$$u(\xi) = \frac{3}{2}\lambda^2\, \text{sech}^2\left(\frac{\lambda}{2}\,\xi + \xi_0\right) - \lambda^2 \,, \tag{74}$$

where $\xi_0 = \tanh^{-1}(\frac{B}{A})$. The solutions (71)-(74) represent the solitary wave solution of the (1+1)-dimensional nonlinear Ostrovsky equation (56).

4 Conclusion

In this paper, we have seen that three types of traveling wave solutions in terms of hyperbolic, trigonometric and rational functions for the nonlinear dispersive equation, the nonlinear Benjamin-Bona Mahony equation, the nonlinear KdV coupled equations and the nonlinear Ostrovsky equation are successfully found out by using the $(\frac{G'}{G})$-expansion method. From our results obtained in this paper, we conclude that the $(\frac{G'}{G})$-expansion method is powerful, effective and convenient. The performance of this method is reliable, simple and gives many new solutions. The $(\frac{G'}{G})$-expansion method has more advantages: It is direct and concise. It is also a standard and computerizable method which allows us to solve complicated nonlinear evolution equations in mathematical physics. Finally, the solutions of the proposed nonlinear evolution equations in this paper have many potential applications in physics and engineering.

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