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Existence of Positive Solutions for a Class of p-Laplacian Systems

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Abstract: We mainly consider the existence of a positive solution of the following system

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda f(v) & in\Omega, \\ -\Delta_p v + |v|^{p-2}v = \lambda g(u) & in\Omega, \\ u = v = 0 & on\partial\Omega, \end{cases}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u), p > 1, \lambda$ is a positive parameter, and $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$. we proved the existence of a positive solution for λ large when

$$\lim_{u\to+\infty} \frac{f[M(g(u))^{\frac{1}{(p-1)}}]}{u^{p-1}} = 0 \quad , \quad for\ every \quad M > 0.$$

Keywords: Nonlinear elliptic equation; p-Laplacian Systems; sub-super solution
AMS subject classification: 35J60, 35B30, 35B40.

1 Introduction

In this work, we study the existence of a positive solution for the system

$$(p) \begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda f(v) & in\Omega, \\ -\Delta_p v + |v|^{p-2}v = \lambda g(u) & in\Omega, \\ u = v = 0 & on\partial\Omega \end{cases}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u), p > 1, \lambda$ is a positive parameter, and $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$.

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Our motivation comes from [8], where the authors considered the existence of positive solutions for nonlinear elliptic equation $-\Delta u + u = f(x, u)$ in a bounded smooth domain $\Omega \subset R^N$ with a nonlinear boundary value condition. The existence results are obtained by the sub-supersolution method and the Mountain pass Lemma. And nonexistence is also considered.

In [6], the authors consider the existence of positive weak solutions for the following p-Laplacian problems

$$(I)\begin{cases} -\Delta_p u = \lambda f(v) & in\Omega, \\ -\Delta_p v = \lambda g(u) & in\Omega, \\ u = v = 0 & on\partial\Omega \end{cases}$$

the first eigenfunction is used to construct the subsolution of p-Laplacian problems successfully.On the condition that λ is large enough and

$$\lim_{u\to+\infty} \frac{f[M(g(u))^{\frac{1}{(p-1)}}]}{u^{p-1}} = 0 \quad , \quad for\ every \quad M > 0,$$

the authors give the existence of positive solutions for problem (I).

In [3], the author consider the existence and nonexistence of positive weak solution to the following quasilinear elliptic system:

$$(II)\begin{cases} -\Delta_p u = \lambda f(u, v) = \lambda u^\alpha v^\gamma & in\Omega, \\ -\Delta_q v = \lambda g(u, v) = \lambda u^\delta v^\beta & in\Omega, \\ u = v = 0 & on\partial\Omega \end{cases}$$

the first eigenfunction is used to construct the subsolution of problem(II), the main results are as following

$(i)\ If\ \alpha, \beta \geq 0, \gamma, \delta > 0, \theta = (p-1-\alpha)(q-1-\beta) - \gamma\delta > 0$, then problem (II) has a positive weak solution for each $\lambda > 0$;

$(ii)\ If\ \theta = 0\ and\ p\gamma = q(p-1-\alpha)$, then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, then problem (II) has no nontrivial nonnegative weak solution.

In this paper, we consider the existence of a positive solution of the problem (p) based on the method of sub-supersolutions. First we give the following hypotheses:

(H_1) $\Omega \subset R^N$ is an open bounded domain with smooth boundary $\partial\Omega$.

(H_2) $f, g : [0, +\infty] \to R^+ \cup \{0\}$ are C^1, monotone functions such that

$$\lim_{u\to+\infty} f(u) = \lim_{u\to+\infty} g(u) = +\infty$$

(H_3) $(f + g)(0)$ is not identically zero.

(H_4) For any positive constant M

$$\lim_{u\to+\infty} \frac{f[M(g(u))^{\frac{1}{(p-1)}}]}{u^{p-1}} = 0$$

Let $W^{1,p}(\Omega) = \{u \in L^p(\Omega) : |\nabla u| \in L^p(\Omega)\}$ with the norm $\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega (|\nabla u|^p + |u|^p) dx\right)^{\frac{1}{p}}$; then $W^{1,p}(\Omega)$ is a Banach space.

Definition 1.1 *If $u, v \in W^{1,p}(\Omega)$, (u, v) is called a weak solution of the problem (p) if it satisfies*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla q dx + \int_\Omega |u|^{p-2}uq dx = \lambda \int_\Omega f(v)q dx, \quad \forall\, q \in W_0^{1,p}(\Omega),$$

and

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla q dx + \int_\Omega |v|^{p-2}vq dx = \lambda \int_\Omega g(u)q dx, \quad \forall\, q \in W_0^{1,p}(\Omega).$$

2 Existence results

Theorem 2.1 *Let $H_1 - H_4$ hold. Then (p) has one positive solution (u, v).*

Proof. By using a method of [6] we shall establish Theorem 2.1 by constructing a subsolution (Φ_1, Φ_2) and supersolution (z_1, z_2) of (p), such that $\Phi_1 \le z_1$ and $\Phi_2 \le z_2$. That is (Φ_1, Φ_2) and (z_1, z_2) satisfies

$$\begin{cases} \int_\Omega |\nabla \Phi_1|^{p-2}\nabla \Phi_1 \cdot \nabla q dx + \int_\Omega |\Phi_1|^{p-2}\Phi_1 q dx \le \lambda \int_\Omega f(\Phi_2)q dx, \\ \int_\Omega |\nabla \Phi_2|^{p-2}\nabla \Phi_2 \cdot \nabla q dx + \int_\Omega |\Phi_2|^{p-2}\Phi_2 q dx \le \lambda \int_\Omega g(\Phi_1)q dx, \end{cases}$$

and

$$\begin{cases} \int_\Omega |\nabla z_1|^{p-2}\nabla z_1 \cdot \nabla q dx + \int_\Omega |z_1|^{p-2}z_1 q dx \ge \lambda \int_\Omega f(z_2)q dx, \\ \int_\Omega |\nabla z_2|^{p-2}\nabla z_2 \cdot \nabla q dx + \int_\Omega |z_2|^{p-2}z_2 q dx \ge \lambda \int_\Omega g(z_1)q dx, \end{cases}$$

for all $q \in W_0^{1,p}(\Omega)$ with $q \ge 0$.

$(0, 0)$ is a subsolution of the problem (p), and $(0, 0)$ is not a solution of the problem (p) by (H_3). We construct a supersolution of (p).

Let Φ_0 be the solution of

$$-\Delta_p \Phi_0 + |\Phi_0|^{p-2}\phi_0 = 1 \quad in\, \Omega, \quad \Phi_0 = 0 \quad on\, \partial\Omega.$$

Let

$$(z_1, z_2) = \left(\frac{C}{\mu}\lambda^{\frac{1}{(p-1)}}\Phi_0, [g(C\lambda^{\frac{1}{(p-1)}})]^{\frac{1}{(p-1)}}\lambda^{\frac{1}{(p-1)}}\Phi_0\right),$$

where $\mu = \|\Phi_0\|_\infty$ and $C > 0$ is a Large number to be chosen later. we shall verify that (z_1, z_2) is a supersolution of (p) for λ large. To this end, let $q \in W_0^{1,p}(\Omega)$ with $q \ge 0$. Then we have

$$\begin{aligned} \int_\Omega |\nabla z_1|^{p-2}\nabla z_1 \cdot \nabla q dx + \int_\Omega |z_1|^{p-2}z_1 q dx &= \lambda(\frac{C}{\mu})^{p-1}\int_\Omega |\nabla \phi_0|^{p-2}\nabla \phi_0 \cdot \nabla q dx + \\ \lambda(\frac{C}{\mu})^{p-1}\int_\Omega |\phi_0|^{p-2}\phi_0 q dx &= \lambda(\frac{C}{\mu})^{p-1}\left(\int_\Omega |\nabla \phi_0|^{p-2}\nabla \phi_0 \cdot \nabla q dx + \int_\Omega |\phi_0|^{p-2}\phi_0 q dx\right) \\ &= \lambda(\frac{C}{\mu})^{p-1}\int_\Omega q dx. \end{aligned}$$

By (H_4), we can choose C Large enough so that

$$(C\lambda^{\frac{1}{(p-1)}})^{p-1} \geq (\mu^{p-1}\lambda) f([\lambda^{\frac{1}{(p-1)}}\mu][g(C\lambda^{\frac{1}{(p-1)}})]^{\frac{1}{(p-1)}},$$

and therefore

$$\begin{aligned}\int_\Omega |\nabla z_1|^{p-2}\nabla z_1 \cdot \nabla q dx + \int_\Omega |z_1|^{p-2} z_1 q dx &\geq \lambda \int_\Omega f([\lambda^{\frac{1}{(p-1)}}\mu][g(C\lambda^{\frac{1}{(p-1)}})]^{\frac{1}{(p-1)}} q dx \\ &\geq \lambda \int_\Omega f(z_2) q dx.\end{aligned}$$

Next,

$$\begin{aligned}\int_\Omega |\nabla z_2|^{p-2}\nabla z_2 \cdot \nabla q dx + \int_\Omega |z_2|^{p-2} z_2 q dx &= \lambda g(C\lambda^{\frac{1}{(p-1)}}) \int_\Omega |\nabla \phi_0|^{p-2}\nabla \phi_0 \cdot \nabla q dx + \\ &\quad \lambda g(C\lambda^{\frac{1}{(p-1)}}) \int_\Omega |\phi_0|^{p-2}\phi_0 q dx \\ &= \lambda g(C\lambda^{\frac{1}{(p-1)}}) \left(\int_\Omega |\nabla \phi_0|^{p-2}\nabla \phi_0 \cdot \nabla q dx + \int_\Omega |\phi_0|^{p-2}\phi_0 q dx\right) \\ &= \lambda g(C\lambda^{\frac{1}{(p-1)}}) \int_\Omega q dx \geq \lambda \int_\Omega g(C\mu^{-1}\lambda^{\frac{1}{(p-1)}}\Phi_0) q dx \\ &= \lambda \int_\Omega g(z_1) q dx,\end{aligned}$$

i.e. (z_1, z_2) is a supersolution of (p) with $z_i \geq 0$ for C Large, $i = 1, 2$. Thus, there exists a solution (u, v) of (p) with $0 \leq u \leq z_1$, $0 \leq v \leq z_2$. This completes the proof. ■

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