

## Existence and Uniqueness of Neutral Functional Differential Equations with Random Impulses

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**Abstract:**In this paper, the existence and uniqueness of the solution of random impulsive neutral functional differential equation is investigated under sufficient condition. The results are obtained by using the Contraction principle.

**Keywords:**neutral functional differential equation; random impulse; existence; uniqueness

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### 1 Introduction

Many evolution processes from fields as diverse as physics, population dynamics, aeronautics, economics, telecommunications and engineering are characterized by the fact that they undergo abrupt change of state at certain moments of time between intervals of continuous evolution. The duration of these changes are often negligible compared to the total duration of process act instantaneously in the form of impulses. It is now being recognized that the theory of impulsive differential equations is not only richer than the corresponding theory of differential equations but also represents a more natural frame work for mathematical modeling of many real world phenomena see [3, 4, 7, 8, 10, 11] and reference therein.

The impulses are exist at fixed time or at random time ie., they are deterministic or random. There are lot of papers which investigate the qualitative properties of fixed type impulses see [3, 4, 7, 8, 10, 11], but there are few papers which studies the random type impulses. Wu and Meng [12], first brought forward random impulsive ordinary differential equations and investigated boundedness of solutions to these models by Liapunov's direct function. Wu et al. [13–16], have been studied some qualitative properties of differential equations with random impulses.

In [3, 4, 7], the authors study the impulsive neutral type of differential equations at fixed points. In [9, 17], the authors studied the existence results under measure of noncompactness. Afrouzi et. al [1, 2], studied the existence of numerical methods for multiple solutions of Dirichlet boundary value problems. In this paper, we investigate a neutral type of differential equation which was first studied by Dhage [5, 6], in Banach algebra. The physical situations for the problem studied in [5] and in this paper occurs are not known so far. Therefore its importance in applications is yet to be investigated. So, the problem under study is new to the literature and so are the existence results to the theory of nonlinear problems of ordinary differential equation.

The paper is organized as follows: Some preliminaries are presented in section 2. In section 3, we investigate the existence of solution of neutral differential equations with random impulses by using Contraction mapping principal and finally in section 4, we present an example for the system studied.

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## 2 Preliminaries

Let  $\mathfrak{R}^n$  be the  $n$ -dimensional Euclidean space and  $\Omega$  a nonempty set. Assume that  $\tau_k$  is a random variable defined from  $\Omega$  to  $D_k \stackrel{def.}{=} (0, d_k)$  for all  $k = 1, 2, \dots$ , where  $0 < d_k < +\infty$ . Furthermore, assume that  $\tau_i$  and  $\tau_j$  are independent with each other as  $i \neq j$  for  $i, j = 1, 2, \dots$ . Let  $\tau \in \mathfrak{R}$  be a constant. For the sake of simplicity, we denote

$$\mathfrak{R}_\tau = [\tau, T], \quad \mathfrak{R}^+ = [0, +\infty).$$

We consider neutral functional differential equations with random impulses of the form

$$\begin{cases} \left[ \frac{x(t)}{g(t, x_t)} \right]' = f(t, x_t), & t \neq \xi_k, \quad t \in [\tau, T], \\ x(\xi_k) = b_k(\tau_k)x(\xi_k^-), & k = 1, 2, \dots, \\ x_{t_0} = \varphi, \end{cases} \tag{2.1}$$

where the functional  $g : \mathfrak{R}_\tau \times C \rightarrow \mathfrak{R}^n - \{0\}$ ,  $f : \mathfrak{R}_\tau \times C \rightarrow \mathfrak{R}^n$ , for each  $\gamma > 0$ , we define  $C(\gamma) = \{\zeta \in C : \|\zeta\|^2 \leq \gamma\}$ ,  $C = C([-r, 0], \mathfrak{R}^n)$  is the set of piecewise continuous functions mapping  $[-r, 0]$  into  $\mathfrak{R}^n$  with some given  $r > 0$ ;  $x_t$  is a function when  $t$  is fixed, defined by  $x_t(s) = x(t + s)$  for all  $s \in [-r, 0]$ ;  $\xi_0 = t_0$  and  $\xi_k = \xi_{k-1} + \tau_k$  for  $k = 1, 2, \dots$ , here  $t_0 \in \mathfrak{R}_\tau$  is arbitrary given real number. Obviously,  $t_0 = \xi_0 < \xi_1 < \xi_2 < \dots < \xi_k < \dots$ ;  $b_k : D_k \rightarrow \mathfrak{R}^{n \times n}$  is a matrix-valued function for each  $k = 1, 2, \dots$ ;  $x(\xi_k^-) = \lim_{t \uparrow \xi_k} x(t)$  according to their paths with the norm  $\|x\|_t = \sup_{t-r \leq s \leq t} |x(s)|$  for each  $t$  satisfying  $\tau \leq t \leq T$  and  $T$  is a given number satisfying  $\tau < T$ ;  $\varphi$  is a function defined from  $[-r, 0]$  to  $\mathfrak{R}^n$ .

Denote  $\{B_t, t \geq 0\}$  the simple counting process generated by  $\{\xi_n\}$ , that is,  $\{B_t \geq n\} = \{\xi_n \leq t\}$ , and denote  $\mathcal{F}_t$  the  $\sigma$ -algebra generated by  $\{B_t, t \geq 0\}$ . Then  $(\Omega, P, \{\mathcal{F}_t\})$  is a probability space. For the simplification, denote  $\Phi_T \stackrel{def.}{=} \Phi_T(\psi)$  the Banach space with norm  $\|\psi\|_{\Phi_T} = (E\|\psi\|_t^2)^{1/2}$ , where  $\psi$  is any function defined from  $[\tau - r, T]$  to  $\mathfrak{R}^n$ .

In the next section, we will discuss the existence and uniqueness of system (2.1) in  $(\Omega, P, \{\mathcal{F}_t\})$ .

**Definition 2.1** A stochastic process  $\{x(t), t_0 - r \leq t \leq T\}$  is called a solution to equation (2.1) in  $(\Omega, P, \{\mathcal{F}_t\})$ , if

- (i)  $x(t)$  is  $\mathcal{F}_t$ -adapted for  $t \geq t_0$ ;
- (ii)  $x(t_0 + s) = \varphi(s)$  when  $s \in [-r, 0]$ , and

$$\begin{aligned} x(t) = & \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^k b_i(\tau_i) g(t, x_t) \left[ \frac{\varphi(0)}{g(0, \varphi(0))} \right] + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) g(t, x_t) \int_{\xi_{i-1}}^{\xi_i} f(s, x_s) ds \right. \\ & \left. + g(t, x_t) \int_{\xi_k}^t f(s, x_s) ds \right] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in [t_0, T], \end{aligned} \tag{2.2}$$

where  $\prod_{j=i}^k b_j(\tau_j) = b_k(\tau_k) b_{k-1}(\tau_{k-1}) \dots b_i(\tau_i)$ ,  $\prod_{j=m}^n (\cdot) = 1$  as  $m > n$  and  $I_A(\cdot)$  is the index function, i.e.,

$$I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}$$

- (iii)  $\frac{x(t)}{g(t, x_t)}$  is differentiable and satisfy the equation (2.1) for  $t \in [\tau, T]$ .

## 3 Existence and uniqueness by contraction mapping

In this section, we discuss the existence and uniqueness of solutions of system (2.1). Before give the main results, we introduce the following hypotheses.

$(H_1)$  : The functions  $g$  and  $f$  satisfies the Lipschitz condition and there exists positive constants  $L_1, L_2 > 0$  for  $\psi, \zeta \in C$  and  $t \in [\tau, T]$  such that

$$\begin{aligned} \|g(t, \psi_t) - g(t, \zeta_t)\|^2 & \leq L_1 \|\psi - \zeta\|_t^2 \\ \|f(t, \psi_t) - f(t, \zeta_t)\|^2 & \leq L_2 \|\psi - \zeta\|_t^2. \end{aligned}$$

(H<sub>2</sub>) : The functions  $g : \mathfrak{R}_\tau \times C \rightarrow \mathfrak{R}^n - \{0\}$ ,  $f : \mathfrak{R}_\tau \times C \rightarrow \mathfrak{R}^n$  are continuous and there exists a non-negative constant  $\kappa$  such that

$$\begin{aligned} \|g(t, 0)\|^2 &\leq \kappa \\ \|f(t, 0)\|^2 &\leq \kappa. \end{aligned}$$

(H<sub>3</sub>) There exists a constants  $0 \leq \gamma_1 < 1$  and  $\gamma_2 > 0$  for  $\varphi(0) \in C$  such that  $\|g(0, \varphi(0))\|^2 \leq \gamma_1 \|\varphi(0)\|^2 + \gamma_2$

$$(H_4) : E \left[ \max_{i,k} \left\{ \prod_{j=i}^k \|b_j(\tau_j)\|^2 \right\} \right] < \infty.$$

**Theorem 3.1** Let the hypotheses (H<sub>1</sub>)–(H<sub>4</sub>) holds, then there exists a constant  $C > 0$  such that

$$E \left[ \max_{i,k} \left\{ \prod_{j=i}^k \|b_j(\tau_j)\|^2 \right\} \right] \leq C,$$

If the inequality

$$\frac{2CL_1E\|\varphi(0)\|^2}{\gamma_1E\|\varphi(0)\|^2+\gamma_2} + 2 \max\{1, C\}(T - t_0)^2 [L_1(L_2\gamma + \kappa) + (L_1\gamma + \kappa)L_2] < 1 \quad (3.1)$$

is satisfied, then the equation (2.1) has a unique solution in  $\Phi_T$ .

**Proof.** Let  $T$  be an arbitrary positive number  $t_0 < T < +\infty$ . To apply the principle, we define the nonlinear operator  $P : \Phi_T \rightarrow \Phi_T$  as follows.

$$(Px)(t) = \varphi(t - t_0), \quad t \in [t_0 - r, t_0],$$

and

$$\begin{aligned} (Px)(t) &= \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^k b_i(\tau_i) g(t, x_t) \frac{\varphi(0)}{g(0, \varphi(0))} + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) g(t, x_t) \int_{\xi_{i-1}}^{\xi_i} f(s, x_s) ds \right. \\ &\quad \left. + g(t, x_t) \int_{\xi_k}^t f(s, x_s) ds \right] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in [t_0, T]. \end{aligned}$$

Now, we have to show that  $P$  maps  $\Phi_T$  into itself. Clearly  $(Px) : \Phi_T \rightarrow \Phi_T$  is continuous with  $(Px)_{t_0} = \varphi$  and

$$\begin{aligned} &\|(Px)(t)\|^2 \\ &= \left\| \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^k b_i(\tau_i) g(t, x_t) \frac{\varphi(0)}{g(0, \varphi(0))} + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) g(t, x_t) \int_{\xi_{i-1}}^{\xi_i} f(s, x_s) ds \right. \right. \\ &\quad \left. \left. + g(t, x_t) \int_{\xi_k}^t f(s, x_s) ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right\|^2 \\ &\leq 2 \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^k \|b_i(\tau_i)\|^2 \|g(t, x_t)\|^2 \left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 I_{[\xi_k, \xi_{k+1})}(t) \right] \\ &\quad + 2 \left[ \sum_{k=0}^{+\infty} \left[ \sum_{i=1}^k \prod_{j=i}^k \|b_j(\tau_j)\| \|g(t, x_t)\| \int_{\xi_{i-1}}^{\xi_i} \|f(s, x_s)\| ds + \|g(t, x_t)\| \int_{\xi_k}^t \|f(s, x_s)\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \end{aligned}$$

$$\begin{aligned}
 & E\|(Px)(t)\|^2 \\
 & \leq 2E \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^k \|b_i(\tau_i)\|^2 \|g(t, x_t)\|^2 \left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 I_{[\xi_k, \xi_{k+1})}(t) \right] \\
 & \quad + 2E \left[ \sum_{k=0}^{+\infty} \left[ \sum_{i=1}^k \prod_{j=i}^k \|b_j(\tau_j)\| \|g(t, x_t)\| \int_{\xi_{i-1}}^{\xi_i} \|f(s, x_s)\| ds + \|g(t, x_t)\| \int_{\xi_k}^t \|f(s, x_s)\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \\
 & \leq 2E \left[ \max_k \left\{ \prod_{j=i}^k \|b_j(\tau_j)\|^2 \right\} \right] E \|g(t, x_t)\|^2 E \left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 \\
 & \quad + 2E \left[ \max_{i,k} \{1, \prod_{j=i}^k \|b_j(\tau_j)\|^2\} \right] E \left[ \sum_{k=0}^{+\infty} \|g(t, x_t)\| \int_{t_0}^t \|f(s, x_s)\| ds I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \\
 & \leq 2CE \|g(t, x_t)\|^2 E \left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 + 2 \max\{1, C\} E \left[ \|g(t, x_t)\| \int_{t_0}^t \|f(s, x_s)\| ds \right]^2 \\
 & \leq 2CE \|g(t, x_t)\|^2 E \left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 + 2 \max\{1, C\} E \|g(t, x_t)\|^2 (T - t_0) \int_{t_0}^t E \|f(s, x_s)\|^2 ds \\
 & \leq [E \|g(t, x_t)\|^2] \left[ 2CE \left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 + 2 \max\{1, C\} (T - t_0) \int_{t_0}^t E \|f(s, x_s)\|^2 ds \right] \\
 & \leq [E \|g(t, x_t) - g(t, 0)\|^2 + E \|g(t, 0)\|^2] \left[ 2CE \left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 + 4 \max\{1, C\} (T - t_0) \right. \\
 & \quad \left. \times \int_{t_0}^t \{E \|f(s, x_s) - f(s, 0)\|^2 + E \|f(s, 0)\|^2\} ds \right] \\
 & \leq [L_1 E \|x\|_t^2 + \kappa] \left[ 2C \frac{E \|\varphi(0)\|^2}{\gamma_1 E \|\varphi(0)\|^2 + \gamma_2} + 4 \max\{1, C\} (T - t_0) \int_{t_0}^t \{L_2 E \|x\|_s^2 + \kappa\} ds \right] \\
 & \leq [L_1 E \|x\|_t^2 + \kappa] \left[ 2C \frac{E \|\varphi(0)\|^2}{\gamma_1 E \|\varphi(0)\|^2 + \gamma_2} + 4 \max\{1, C\} (T - t_0)^2 \kappa + 4 \max\{1, C\} L_2 (T - t_0) \int_{t_0}^t E \|x\|_s^2 ds \right] \\
 & \leq [L_1 E \|x\|_t^2 + \kappa] [\beta_1 + \beta_2 E \|x\|_t^2]
 \end{aligned}$$

where  $\beta_1 = 2C \frac{E \|\varphi(0)\|^2}{\gamma_1 E \|\varphi(0)\|^2 + \gamma_2} + 4 \max\{1, C\} (T - t_0)^2 \kappa$  and  $\beta_2 = 4 \max\{1, C\} L_2 (T - t_0)^2$

$$\sup_{t_0 \leq t \leq T} E\|(Px)\|_t^2 \leq [L_1 \sup_{t_0 \leq t \leq T} E\|x\|_t^2 + \kappa] [\beta_1 + \beta_2 \sup_{t_0 \leq t \leq T} E\|x\|_t^2] \tag{3.2}$$

from (3.2), we deduce that the operator  $P$  maps  $\Phi_T$  into itself.

To see that  $P$  is a contraction mapping, observe for  $t \in [t_0, T]$ ,

$$\begin{aligned}
 & (Px)(t) - (Py)(t) \\
 & = \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^k b_i(\tau_i) [g(t, x_t) - g(t, y_t)] \left[ \frac{\varphi(0)}{g(0, \varphi(0))} \right] + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) \left\{ g(t, x_t) \int_{\xi_{i-1}}^{\xi_i} f(s, x_s) ds \right. \right. \\
 & \quad \left. \left. - g(t, y_t) \int_{\xi_{i-1}}^{\xi_i} f(s, y_s) ds \right\} + \left\{ g(t, x_t) \int_{\xi_k}^t f(s, x_s) ds - g(t, y_t) \int_{\xi_k}^t f(s, y_s) ds \right\} \right] I_{[\xi_k, \xi_{k+1})}(t)
 \end{aligned}$$

$$\begin{aligned}
 & \|(Px)(t) - (Py)(t)\|^2 \\
 & \leq 2 \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^k \|b_i(\tau_i)\|^2 \|g(t, x_t) - g(t, y_t)\|^2 \left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 I_{[\xi_k, \xi_{k+1})}(t) \right. \\
 & \quad + 2 \left[ \sum_{k=0}^{+\infty} \left[ \sum_{i=1}^k \prod_{j=i}^k \|b_j(\tau_j)\| \|g(t, x_t)\| \int_{\xi_{i-1}}^{\xi_i} \|f(s, x_s)\| ds - g(t, y_t) \int_{\xi_{i-1}}^{\xi_i} \|f(s, y_s)\| ds \right] \right. \\
 & \quad \left. + \|g(t, x_t) \int_{\xi_k}^t f(s, x_s) ds - g(t, y_t) \int_{\xi_k}^t f(s, y_s) ds\| \right] I_{[\xi_k, \xi_{k+1})}(t) \left. \right]^2
 \end{aligned}$$

$$\begin{aligned}
& E\|(Px)(t) - (Py)(t)\|^2 \\
& \leq 2E \left[ \max_k \left\{ \prod_{j=i}^k \|b_j(\tau_j)\|^2 \right\} \right] E\|g(t, x_t) - g(t, y_t)\|^2 E\left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 \\
& \quad + 2E \left[ \sum_{k=0}^{+\infty} \left[ \sum_{i=1}^k \prod_{j=i}^k \|b_j(\tau_j)\| \left[ \|g(t, x_t) \int_{\xi_{i-1}}^{\xi_i} f(s, x_s) ds - g(t, y_t) \int_{\xi_{i-1}}^{\xi_i} f(s, x_s) ds \right. \right. \right. \\
& \quad \left. \left. \left. + \|g(t, y_t) \int_{\xi_{i-1}}^{\xi_i} f(s, x_s) ds - g(t, y_t) \int_{\xi_{i-1}}^{\xi_i} f(s, y_s) ds \right\| \right] + \left[ \|g(t, x_t) \int_{\xi_k}^t f(s, x_s) ds - g(t, y_t) \right. \right. \\
& \quad \left. \left. \times \int_{\xi_k}^t f(s, x_s) ds + \|g(t, y_t) \int_{\xi_k}^t f(s, x_s) ds - g(t, y_t) \int_{\xi_k}^t f(s, y_s) ds \right\| \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \\
& \leq 2CE\|g(t, x_t) - g(t, y_t)\|^2 E\left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 + 2E \left[ \sum_{k=0}^{+\infty} \left[ \sum_{i=1}^k \prod_{j=i}^k \|b_j(\tau_j)\| \left[ \|g(t, x_t) - g(t, y_t)\| \right. \right. \right. \\
& \quad \left. \left. \left. \times \int_{\xi_{i-1}}^{\xi_i} \|f(s, x_s)\| ds + \|g(t, y_t)\| \int_{\xi_{i-1}}^{\xi_i} \|f(s, x_s) - f(s, y_s)\| ds \right] \right] \right. \\
& \quad \left. + \left[ \|g(t, x_t) - g(t, y_t)\| \int_{\xi_k}^t \|f(s, x_s)\| ds + \|g(t, y_t)\| \int_{\xi_k}^t \|f(s, x_s) - f(s, y_s)\| ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \\
& \leq 2CE\|g(t, x_t) - g(t, y_t)\|^2 E\left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 + 2 \max\{1, C\} E \left[ \sum_{k=0}^{+\infty} \left( \|g(t, x_t) - g(t, y_t)\| \right. \right. \\
& \quad \left. \left. \times \int_{t_0}^t \|f(s, x_s)\| ds + \|g(t, y_t)\| \int_{t_0}^t \|f(s, x_s) - f(s, y_s)\| ds \right) I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \\
& \leq 2CE\|g(t, x_t) - g(t, y_t)\|^2 E\left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 + \max\{1, C\} \left( E\|g(t, x_t) - g(t, y_t)\|^2 (T - t_0) \right. \\
& \quad \left. \times \int_{t_0}^t E\|f(s, x_s)\|^2 ds + E\|g(t, y_t)\|^2 (T - t_0) \int_{t_0}^t E\|f(s, x_s) - f(s, y_s)\|^2 ds \right) \\
& \leq 2CL_1 E\left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 E\|x - y\|_t^2 + \max\{1, C\} (T - t_0) \\
& \quad \times \left[ L_1 E\|x - y\|_t^2 \int_{t_0}^t (L_2 E\|x\|_s^2 + \kappa) ds + (L_1 E\|x\|_t^2 + \kappa) L_2 \int_{t_0}^t E\|x - y\|_s^2 ds \right].
\end{aligned}$$

Then,

$$\begin{aligned}
& \sup_{t_0 \leq t \leq T} E\|(Px) - (Py)\|_t^2 \\
& \leq 2CL_1 E\left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 \sup_{t_0 \leq t \leq T} E\|x - y\|_t^2 + 2 \max\{1, C\} (T - t_0) \left[ L_1 \sup_{t_0 \leq t \leq T} E\|x - y\|_t^2 (T - t_0) \right. \\
& \quad \left. \times (L_2 \sup_{t_0 \leq t \leq T} E\|x\|_t^2 + \kappa) + (L_1 \sup_{t_0 \leq t \leq T} E\|x\|_t^2 + \kappa) L_2 (T - t_0) \sup_{t_0 \leq t \leq T} E\|x - y\|_t^2 \right] \\
& \leq 2CL_1 E\left\| \frac{\varphi(0)}{g(0, \varphi(0))} \right\|^2 \sup_{t_0 \leq t \leq T} E\|x - y\|_t^2 + 2 \max\{1, C\} (T - t_0) \left[ L_1 \sup_{t_0 \leq t \leq T} E\|x - y\|_t^2 (T - t_0) \right. \\
& \quad \left. \times (L_2 \gamma + \kappa) + (L_1 \gamma + \kappa) L_2 (T - t_0) \sup_{t_0 \leq t \leq T} E\|x - y\|_t^2 \right] \\
& \leq \left\{ 2CL_1 \frac{E\|\varphi(0)\|^2}{\gamma_1 E\|\varphi(0)\|^2 + \gamma_2} + 2 \max\{1, C\} (T - t_0)^2 [L_1 (L_2 \gamma + \kappa) + (L_1 \gamma + \kappa) L_2] \right\} \sup_{t_0 \leq t \leq T} E\|x - y\|_t^2
\end{aligned}$$

It results that

$$\begin{aligned}
& \|(Px) - (Py)\|_{\Phi_T}^2 \\
& \leq \left\{ 2CL_1 \frac{E\|\varphi(0)\|^2}{\gamma_1 E\|\varphi(0)\|^2 + \gamma_2} + 2 \max\{1, C\} (T - t_0)^2 [L_1 (L_2 \gamma + \kappa) + (L_1 \gamma + \kappa) L_2] \right\} \|x - y\|^2
\end{aligned}$$

Therefore by (3.1),  $P$  is a contraction mapping. Thus the mapping  $P$  has a unique fixed point  $x$  in  $\Phi_T$  which is a solution of (2.1) with  $\varphi \in C(\gamma)$ . For any  $t \in [t_0, T]$  and  $t \neq \xi_k$ , there exists  $(\xi_k, \xi_{k+1})$  such that  $t \in (\xi_k, \xi_{k+1})$ , which yields

$$x(t) = \prod_{i=1}^k b_i(\tau_i) g(t, x_t) \left[ \frac{\varphi(0)}{g(0, \varphi(0))} \right] + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) g(t, x_t) \int_{\xi_{i-1}}^{\xi_i} f(s, x_s) ds + g(t, x_t) \int_{\xi_k}^t f(s, x_s) ds$$

and therefore,  $\left[ \frac{x(t)}{g(t, x_t)} \right]' = f(t, x_t)$ , ie.,  $\left[ \frac{x(t)}{g(t, x_t)} \right]' = f(t, x_t)$ , as  $t \neq \xi_k$ . Furthermore,

$$x_{\xi_k} = \prod_{i=1}^k b_i(\tau_i) g(t, x_t) \left[ \frac{\varphi(0)}{g(0, \varphi(0))} \right] + \sum_{i=1}^k \prod_{j=i}^k b_j(\tau_j) g(t, x_t) \int_{\xi_{i-1}}^{\xi_i} f(s, x_s) ds$$

and

$$x_{\xi_k}^- = \prod_{i=1}^{k-1} b_i(\tau_i)g(t, x_t) \left[ \frac{\varphi(0)}{g(0, \varphi(0))} \right] + \sum_{i=1}^{k-1} \prod_{j=i}^{k-1} b_j(\tau_j)g(t, x_t) \int_{\xi_{i-1}}^{\xi_i} f(s, x_s)ds + g(t, x_t) \int_{\xi_{k-1}}^t f(s, x_s)ds$$

imply that  $x_{\xi_k} = b_k(\tau_k)x_{\xi_k}^-$ . Thus,  $x(t)$  is a solution to system (2.1). ■

### 4 An Application:

In this section, we present an example to show value of (2.1) and an application for the obtained result of Theorem 3.1.

**Example 1** Let  $\tau_k$  be a random variable defined in  $D_k \equiv (0, d_k)$  for all  $k = 1, 2, \dots$ , where  $0 < d_k < +\infty$ . Further more, assume that  $\tau_i$  and  $\tau_j$  be independent with each other as  $i \neq j$  for  $i, j = 1, 2, \dots$ .

$$\begin{cases} \left[ \frac{x(t)}{5(1-t) + \frac{\ln(2+t)}{2(1+t)} \sin x(t/2)} \right]' &= \operatorname{argtg} \left[ \frac{t}{t^2+1} \exp(t/2) \right] + \frac{x \sin x}{3+|\sin x|}, \quad t \neq \xi_k, \quad t \in [t_0, T], \\ x(\xi_k) &= p(k)(\tau_k)x(\xi_k^-), \quad k = 1, 2, \dots, \\ x_{t_0} &= \varphi, \end{cases} \tag{4.1}$$

Observe that the problem (4.1) is a special case of the problem (2.1), where the functions  $f$  and  $g$  has the form

$$\begin{aligned} g(t, x_t) &= 5(1-t) + \frac{\ln(2+t)}{2(1+t)} \sin x(t/2) \\ f(t, x_t) &= \operatorname{argtg} \left[ \frac{t}{t^2+1} \exp(t/2) \right] + \frac{x \sin x}{3+|\sin(x)|} \\ b_k(\tau_k) &= p(k)(\tau_k). \end{aligned}$$

It is easily seen that the functions  $f$  and  $g$  satisfies the assumptions  $(H_1)$  with  $L_1 = 1/2$  and  $L_2 = 1/3$ . On the other hand it satisfies  $(H_2)$  with

$$\begin{aligned} \|g(t, 0)\|^2 &\leq 5(1-t) \leq \kappa_1 \\ \|f(t, 0)\|^2 &\leq \left\| \operatorname{argtg} \left[ \frac{t}{t^2+1} \exp(t/2) \right] \right\|^2 \leq \frac{t}{t^2+1} \exp(t/2) \leq \kappa_1 \end{aligned}$$

and it also satisfies  $(H_3)$  that

$$\|g(0, \varphi(0))\|^2 \leq \frac{\ln(2)}{2} \|\varphi(0)\|^2 + 5$$

Moreover the assumption  $(H_4)$  is satisfied. Summing up all the above facts, in view of Theorem 3.1 we conclude that the problem (4.1) has a unique solution  $x = x(t)$  provided that

$$\frac{C E \|\varphi(0)\|^2}{\frac{\ln(2)}{2} E \|\varphi(0)\|^2 + 5} + 2 \max\{1, C\} (T - t_0)^2 \left[ (1/2) ((1/3)\gamma + \kappa_1) + ((1/2)\gamma + \kappa_1) (1/3) \right] < 1.$$

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