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A Solution Form of a Class of Rational Difference Equations

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Abstract: We obtain in this paper the expressions of solutions of the following class of difference equation

$$x_{n+1} = \frac{x_{n-11}}{\pm 1 \pm x_{n-3}x_{n-7}x_{n-11}}, \quad n = 0, 1, ...$$

where the initial values $x_{-j},\ (j = 0, 1, ..., 11)$ are arbitrary non zero real numbers.

Keywords: recursive sequence; periodicity; solutions of difference equations
Mathematics Subject Classification: 39A10

1 Introduction

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For some results in this area, for example: Aloqeili [1] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [5–7] investigated the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Cinar et al.[8] studied the solutions and attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{-1 + x_n x_{n-1} x_{n-2} x_{n-3}}.$$

Elabbasy et al. [9] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}.$$

Elabbasy et al. [10] investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

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Elabbasy et al. [11] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1}=\frac{dx_{n-l}x_{n-k}}{cx_{n-s}-b}+a.$$

Karatas [15-16] get the solutions of the difference equations

$$x_{n+1}=\frac{(-1)^n x_{n-4}}{1+(-1)^n x_n x_{n-1}x_{n-2}x_{n-3}x_{n-4}},\ \ x_{n+1}=\frac{ax_{n-(2k+2)}}{-a+\prod_{i=0}^{2k+2}x_{n-i}}.$$

Simsek [19-20] solved the recursive sequences

$$x_{n+1}=\frac{x_{n-11}}{1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}},\ \ x_{n+1}=\frac{x_{n-3}}{1+x_{n-1}}.$$

In [23] Stevic solved the following problem

$$x_{n+1}=\frac{x_{n-1}}{1+x_n}.$$

Other related results on rational difference equations can be found in refs. [2-4], [12–28].

Similar to the references above, in this paper we obtain the solutions of the following nonlinear difference equations

$$x_{n+1}=\frac{x_{n-11}}{\pm 1\pm x_{n-3}x_{n-7}x_{n-11}},\quad n=0,1,... \tag{1}$$

where the initial values $x_{-j},\ (j=0,1,...,11)$ are arbitrary non zero real numbers.

Let I be some interval of real numbers and let

$$f:I^{k+1}\to I$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k},x_{-k+1},...,x_0\in I$, the difference equation

$$x_{n+1}=f(x_n,x_{n-1},...,x_{n-k}),\ \ n=0,1,..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 *A point $\overline{x}\in I$ is called an equilibrium point of Eq.(2) if*

$$\overline{x}=f(\overline{x},\overline{x},...,\overline{x}).$$

That is, $x_n=\overline{x}$ for $n\geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2 *(Periodicity)*

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p}=x_n$ for all $n\geq -k$.

2 First Equation $x_{n+1}=\dfrac{x_{n-11}}{1+x_{n-3}x_{n-7}x_{n-11}}$

In this section we give a specific form of the solution of the first equation in the form

$$x_{n+1}=\frac{x_{n-11}}{1+x_{n-3}x_{n-7}x_{n-11}},\quad n=0,1,... \tag{3}$$

where the initial values are arbitrary non zero real numbers.

Theorem 1 *Let* $\{x_n\}_{n=-11}^{\infty}$ *be a solution of Eq.(3). Then for* $n = 0, 1, \ldots$

$$\begin{aligned}
x_{12n-11} &= p\prod_{i=0}^{n-1}\left(\frac{1+3iphd}{1+(3i+1)\,phd}\right), & x_{12n-10} &= m\prod_{i=0}^{n-1}\left(\frac{1+3imgc}{1+(3i+1)\,mgc}\right),\\
x_{12n-9} &= l\prod_{i=0}^{n-1}\left(\frac{1+3ilfb}{1+(3i+1)\,lfb}\right), & x_{12n-8} &= k\prod_{i=0}^{n-1}\left(\frac{1+3ikea}{1+(3i+1)\,kea}\right),\\
x_{12n-7} &= h\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)\,phd}{1+(3i+2)\,phd}\right), & x_{12n-6} &= g\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)\,mgc}{1+(3i+2)\,mgc}\right),
\end{aligned}$$

$$\begin{aligned}
x_{12n-5} &= f\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)lfb}{1+(3i+2)\,lfb}\right), & x_{12n-4} &= e\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)kea}{1+(3i+2)\,kea}\right),\\
x_{12n-3} &= d\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)phd}{1+(3i+3)\,phd}\right), & x_{12n-2} &= c\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)mgc}{1+(3i+3)\,mgc}\right),\\
x_{12n-1} &= b\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)lfb}{1+(3i+3)\,lfb}\right), & x_{12n} &= a\prod_{i=0}^{n-1}\left(\frac{1+(3i+2)kea}{1+(3i+3)\,kea}\right),
\end{aligned}$$

where $x_{-11} = p,\ x_{-10} = m,\ x_{-9} = l,\ x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a.$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is:

$$\begin{aligned}
x_{12n-23} &= p\prod_{i=0}^{n-2}\left(\frac{1+3iphd}{1+(3i+1)\,phd}\right), & x_{12n-22} &= m\prod_{i=0}^{n-2}\left(\frac{1+3imgc}{1+(3i+1)\,mgc}\right),\\
x_{12n-21} &= l\prod_{i=0}^{n-2}\left(\frac{1+3ilfb}{1+(3i+1)\,lfb}\right), & x_{12n-20} &= k\prod_{i=0}^{n-2}\left(\frac{1+3ikea}{1+(3i+1)\,kea}\right),\\
x_{12n-19} &= h\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)\,phd}{1+(3i+2)\,phd}\right), & x_{12n-18} &= g\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)\,mgc}{1+(3i+2)\,mgc}\right),
\end{aligned}$$

$$\begin{aligned}
x_{12n-17} &= f\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)lfb}{1+(3i+2)\,lfb}\right), & x_{12n-16} &= e\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)kea}{1+(3i+2)\,kea}\right),\\
x_{12n-15} &= d\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)phd}{1+(3i+3)\,phd}\right), & x_{12n-14} &= c\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)mgc}{1+(3i+3)\,mgc}\right),\\
x_{12n-13} &= b\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)lfb}{1+(3i+3)\,lfb}\right), & x_{12n-12} &= a\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)kea}{1+(3i+3)\,kea}\right).
\end{aligned}$$

Now, it follows from Eq.(3) that

$$\begin{aligned}
x_{12n-11} &= \frac{x_{12n-23}}{1+x_{12n-15}x_{12n-19}x_{12n-23}}\\
&= \frac{p\prod_{i=0}^{n-2}\left(\frac{1+3iphd}{1+(3i+1)\,phd}\right)}{1+d\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)phd}{1+(3i+3)phd}\right)h\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)phd}{1+(3i+2)phd}\right)p\prod_{i=0}^{n-2}\left(\frac{1+3iphd}{1+(3i+1)phd}\right)}
\end{aligned}$$

$$= \frac{p\prod_{i=0}^{n-2}\left(\frac{1+3iphd}{1+(3i+1)\,phd}\right)}{1+\left(\frac{phd}{1+(3n-3)\,phd}\right)} = \frac{p\prod_{i=0}^{n-2}\left(\frac{1+3iphd}{1+(3i+1)\,phd}\right)}{1+\left(\frac{phd}{1+(3n-3)\,phd}\right)}\left(\frac{1+(3n-3)\,phd}{1+(3n-3)\,phd}\right)$$

$$= \frac{p\prod_{i=0}^{n-2}\left(\frac{1+3iphd}{1+(3i+1)\,phd}\right)(1+(3n-3)\,phd)}{1+(3n-2)\,phd}$$

Hence, we have

$$x_{12n-11} = p\prod_{i=0}^{n-1}\left(\frac{1+3iphd}{1+(3i+1)\,phd}\right).$$

Similarly

$$\begin{aligned}
x_{12n-6} &= \frac{x_{12n-18}}{1+x_{12n-10}x_{12n-14}x_{12n-18}} \\
&= \frac{g\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)\,mgc}{1+(3i+2)\,mgc}\right)}{1+m\prod_{i=0}^{n-1}\left(\frac{1+3imgc}{1+(3i+1)mgc}\right)c\prod_{i=0}^{n-2}\left(\frac{1+(3i+2)mgc}{1+(3i+3)mgc}\right)g\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)mgc}{1+(3i+2)mgc}\right)} \\
&= \frac{g\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)\,mgc}{1+(3i+2)\,mgc}\right)}{1+\left(\frac{mgc}{1+(3n-2)\,mgc}\right)} = \frac{g\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)\,mgc}{1+(3i+2)\,mgc}\right)}{\left(\frac{1+(3n-2)\,mgc+mgc}{1+(3n-2)\,mgc}\right)} \\
&= g\prod_{i=0}^{n-2}\left(\frac{1+(3i+1)\,mgc}{1+(3i+2)\,mgc}\right)\left(\frac{1+(3n-2)\,mgc}{1+(3n-1)\,mgc}\right).
\end{aligned}$$

Hence, we have

$$x_{12n-6} = g\prod_{i=0}^{n-1}\left(\frac{1+(3i+1)\,mgc}{1+(3i+2)\,mgc}\right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. ■

Theorem 2 *Eq.(3) has one equilibrium point which is the number zero.*

Proof. For the equilibrium points of Eq.(3), we can write

$$\overline{x} = \frac{\overline{x}}{1+\overline{x}^3}.$$

Then we have

$$\overline{x} + \overline{x}^4 = \overline{x},$$

or,

$$\overline{x}^4 = 0.$$

Thus the equilibrium point of Eq.(3) is $\overline{x} = 0$. ■

Theorem 3 *Every positive solution of Eq.(3) is bounded.*

Proof. Let $\{x_n\}_{n=-11}^{\infty}$ be a solution of Eq.(3). It follows from Eq.(3) that

$$x_{n+1} = \frac{x_{n-11}}{1+x_{n-3}x_{n-7}x_{n-11}} \leq x_{n-11}.$$

Then

$$x_{n+1} \leq x_{n-11} \quad \text{for all} \quad n \geq 0.$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ is decreasing and so is bounded from above by $M = \max\{x_{-11}, x_{-10}, x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}$. ■

Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq. (3).

Example 1. We assume $x_{-11} = 13,\ x_{-10} = 9,\ x_{-9} = 1.8, x_{-8} = 0.7,\ x_{-7} = 0.4,\ x_{-6} = 0.2,\ x_{-5} = 1.3,\ x_{-4} = 6, x_{-3} = 0.2,\ x_{-2} = 4,\ x_{-1} = 0.2,\ x_0 = 4$ See Fig. 1.

Example 2. See Fig. 2, since $x_{-11} = 3,\ x_{-10} = 7,\ x_{-9} = 1.8, x_{-8} = 7,\ x_{-7} = -4,\ x_{-6} = 5,\ x_{-5} = -3,\ x_{-4} = 0.6,\ x_{-3} = 9,\ x_{-2} = -4,\ x_{-1} = 2,\ x_0 = 9.$

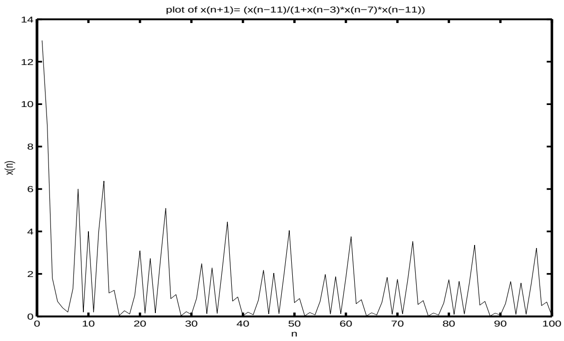

Figure 1: Example 1.

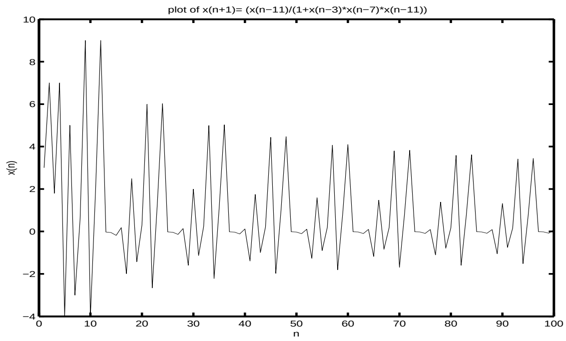

Figure 2: Example 2.

3 Second Equation $x_{n+1}=\dfrac{x_{n-11}}{-1+x_{n-3}x_{n-7}x_{n-11}}$

In this section we obtain the solution of the second equation in the form

$$x_{n+1}=\frac{x_{n-11}}{-1+x_{n-3}x_{n-7}x_{n-11}},\quad n=0,1,... \tag{4}$$

where the initial values are arbitrary non zero real numbers with $x_{-11}x_{-7}x_{-3}\neq 1, x_{-10}x_{-6}x_{-2}\neq 1,\ x_{-9}x_{-5}x_{-1}\neq 1,\ x_{-8}x_{-4}x_0\neq 1.$

Theorem 4 *Let $\{x_n\}_{n=-11}^{\infty}$ be a solution of Eq.(4). Then every solution of Eq.(4) is periodic with period 24 and for $n=0,1,...$*

$$\begin{aligned}
x_{24n-11} &= p, \quad x_{24n-10}=m, \quad x_{24n-9}=l, \quad x_{24n-8}=k,\\
x_{24n-7} &= h, \quad x_{24n-6}=g, \quad x_{24n-5}=f, \quad x_{24n-4}=e,\\
x_{24n-3} &= d, \quad x_{24n-2}=c, \quad x_{24n-1}=b, \quad x_{24n}=a,\\
x_{24n+1} &= \frac{p}{-1+phd}, \quad x_{24n+2}=\frac{m}{-1+mgc}, \quad x_{24n+3}=\frac{l}{-1+lfb},\\
x_{24n+4} &= \frac{k}{-1+kea}, \quad x_{24n+5}=h\left(-1+phd\right), \quad x_{24n+6}=g\left(-1+mgc\right),\\
x_{24n+7} &= f\left(-1+lfb\right), \quad x_{24n+8}=e\left(-1+kea\right), \quad x_{24n+9}=\frac{d}{-1+phd},\\
x_{24n+10} &= \frac{c}{-1+mgc}, \quad x_{24n+11}=\frac{b}{-1+lfb}, \quad x_{24n+12}=\frac{a}{-1+kea},
\end{aligned} \tag{5}$$

where $x_{-11}=p,\ x_{-10}=m,\ x_{-9}=l,\ x_{-8}=k,\ x_{-7}=h,\ x_{-6}=g,\ x_{-5}=f,\ x_{-4}=e,\ x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b,\ x_{-0}=a.$

Proof. For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is:

$$\begin{aligned}
x_{24n-35} &= p, \quad x_{24n-34}=m, \quad x_{24n-33}=l, \quad x_{24n-32}=k,\\
x_{24n-31} &= h, \quad x_{24n-30}=g, \quad x_{24n-29}=f, \quad x_{24n-28}=e,\\
x_{24n-27} &= d, \quad x_{24n-26}=c, \quad x_{24n-25}=b, \quad x_{24n-24}=a,\\
x_{24n-23} &= \frac{p}{-1+phd}, \quad x_{24n-22}=\frac{m}{-1+mgc}, \quad x_{24n-21}=\frac{l}{-1+lfb},\\
x_{24n-20} &= \frac{k}{-1+kea}, \quad x_{24n-19}=h\left(-1+phd\right), \quad x_{24n-18}=g\left(-1+mgc\right),\\
x_{24n-17} &= f\left(-1+lfb\right), \quad x_{24n-16}=e\left(-1+kea\right), \quad x_{24n-15}=\frac{d}{-1+phd},\\
x_{24n-14} &= \frac{c}{-1+mgc}, \quad x_{24n-13}=\frac{b}{-1+lfb}, \quad x_{24n-12}=\frac{a}{-1+kea},
\end{aligned}$$

Now, it follows from Eq.(4) that

$$x_{24n-11} = \frac{x_{24n-23}}{-1+x_{24n-15}x_{24n-19}x_{24n-23}} = \frac{\frac{p}{-1+phd}}{-1+\frac{d}{-1+phd}h\left(-1+phd\right)\frac{p}{-1+phd}}$$
$$= \frac{p}{(-1+phd)\left(-1+\frac{phd}{-1+phd}\right)}.$$

Hence, we have

$$x_{24n-11} = p.$$

Similarly

$$x_{24n+10} = \frac{x_{24n-2}}{-1+x_{24n+6}x_{24n+2}x_{24n-2}} = \frac{c}{-1+g\left(-1+mgc\right)\frac{m}{-1+mgc}c}$$
$$= \frac{c}{-1+mgc}.$$

Similarly, one can easily prove the other relations. Thus, the proof is completed. ■

Theorem 5 *Eq.(4) has two equilibrium points which are* $0, \sqrt[3]{2}$.

Proof. For the equilibrium points of Eq.(4), we can write

$$\overline{x} = \frac{\overline{x}}{-1+\overline{x}^3}.$$

Then we have

$$-\overline{x}+\overline{x}^4 = \overline{x},$$

or,

$$\overline{x}(\overline{x}^3-2) = 0.$$

Thus the equilibrium points of Eq.(4) are $0, \sqrt[3]{2}$. ■

Theorem 6 *Eq.(4) has a periodic solutions of period twelve iff* $phd = mgc = lfb = kea = 2$ *and will be take the form* $\{p, m, l, k, h, g, f, e, d, c, b, a, p, m, l, k, h, g, f, e, d, c, b, a, ...\}$.

Proof. First suppose that there exists a prime period twelve solution

$$p, m, l, k, h, g, f, e, d, c, b, a, p, m, l, k, h, g, f, e, d, c, b, a, ...$$

of Eq.(4), we see from Eq.(5) that

$$p = \frac{p}{-1+phd}, \quad m = \frac{m}{-1+mgc}, \quad l = \frac{l}{-1+lfb}$$

$$k = \frac{k}{-1+kea}, \quad h = h\left(-1+phd\right), \quad g = g\left(-1+mgc\right),$$
$$f = f\left(-1+lfb\right), \quad e = e\left(-1+kea\right), \quad d = \frac{d}{-1+phd},$$
$$c = \frac{c}{-1+mgc}, \quad b = \frac{b}{-1+lfb}, \quad a = \frac{a}{-1+kea},$$

Then

$$phd = mgc = lfb = kea = 2.$$

Second assume that $phd = mgc = lfb = kea = 2$. Then we see from Eq.(5) that

$$\begin{aligned} x_{24n-11} &= p, \quad x_{24n-10} = m, \quad x_{24n-9} = l, \quad x_{24n-8} = k, \\ x_{24n-7} &= h, \quad x_{24n-6} = g, \quad x_{24n-5} = f, \quad x_{24n-4} = e, \\ x_{24n-3} &= d, \quad x_{24n-2} = c, \quad x_{24n-1} = b, \quad x_{24n} = a, \\ x_{24n+1} &= p, \quad x_{24n+2} = m, \quad x_{24n+3} = l, \quad x_{24n+4} = k, \\ x_{24n+5} &= h, \quad x_{24n+6} = g, \quad x_{24n+7} = f, \quad x_{24n+8} = e, \\ x_{24n+9} &= d, \quad x_{24n+10} = c, \quad x_{24n+11} = b, \quad x_{24n+12} = a, \end{aligned}$$

Thus we have a periodic solution of period twelve and the proof is complete. ■

Numerical examples

Here we will represent different types of solutions of Eq. (4).

Example 3. We consider $x_{-11} = 0.8,\ x_{-10} = 0.11,\ x_{-9} = 1.8, x_{-8} = 1.7,\ x_{-7} = -1.4,\ x_{-6} = 0.5,$ $x_{-5} = 0.3,\ x_{-4} = 0.6,\ x_{-3} = 0.9,\ x_{-2} = 4,\ x_{-1} = -2,\ x_0 = -7$. See Fig. 3.

Example 4. See Fig. 4, since $x_{-11} = 0.5,\ x_{-10} = -2,\ x_{-9} = 4, x_{-8} = 7,\ x_{-7} = 3,\ x_{-6} = 1/5,$ $x_{-5} = -3,\ x_{-4} = 3,\ x_{-3} = 4/3,\ x_{-2} = -5,\ x_{-1} = -1/6,\ x_0 = 2/21$.

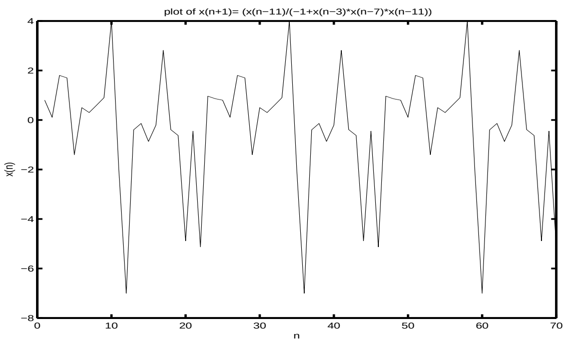

Figure 3: Example 3.

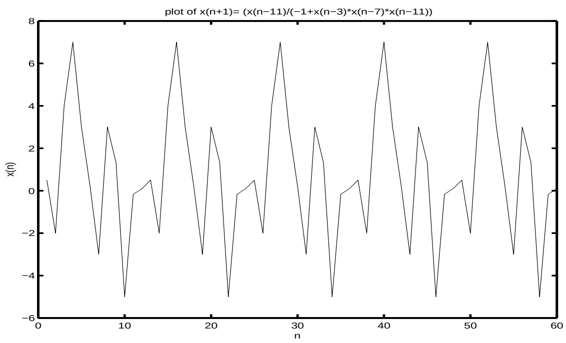

Figure 4: Example 4.

The following cases can be proved similarly.

4 Third Equation $x_{n+1} = \dfrac{x_{n-11}}{1 - x_{n-3}x_{n-7}x_{n-11}}$

In this section we get the solution of the third following equation

$$x_{n+1} = \frac{x_{n-11}}{1 - x_{n-3}x_{n-7}x_{n-11}}, \quad n = 0, 1, ... \tag{6}$$

where the initial values are arbitrary non zero real numbers.

Theorem 7 *Let $\{x_n\}_{n=-11}^{\infty}$ be a solution of Eq.(6). Then for $n = 0, 1, ...$*

$$\begin{aligned} x_{12n-11} &= p\prod_{i=0}^{n-1}\left(\frac{1-3iphd}{1-(3i+1)\,phd}\right), \quad x_{12n-10} = m\prod_{i=0}^{n-1}\left(\frac{1-3imgc}{1+(3i-1)\,mgc}\right), \\ x_{12n-9} &= l\prod_{i=0}^{n-1}\left(\frac{1-3ilfb}{1-(3i+1)\,lfb}\right), \quad x_{12n-8} = k\prod_{i=0}^{n-1}\left(\frac{1-3ikea}{1-(3i+1)\,kea}\right), \\ x_{12n-7} &= h\prod_{i=0}^{n-1}\left(\frac{1-(3i+1)\,phd}{1-(3i+2)\,phd}\right), \quad x_{12n-6} = g\prod_{i=0}^{n-1}\left(\frac{1-(3i+1)\,mgc}{1-(3i+2)\,mgc}\right), \end{aligned}$$

$$
\begin{aligned}
x_{12n-5} &= f\prod_{i=0}^{n-1}\left(\frac{1-(3i+1)lfb}{1-(3i+2)\,lfb}\right), \qquad x_{12n-4}=e\prod_{i=0}^{n-1}\left(\frac{1-(3i+1)kea}{1-(3i+2)\,kea}\right),\\
x_{12n-3} &= d\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)phd}{1-(3i+3)\,phd}\right), \qquad x_{12n-2}=c\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)mgc}{1-(3i+3)\,mgc}\right),\\
x_{12n-1} &= b\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)lfb}{1-(3i+3)\,lfb}\right), \qquad x_{12n}=a\prod_{i=0}^{n-1}\left(\frac{1-(3i+2)kea}{1-(3i+3)\,kea}\right),
\end{aligned}
$$

where $x_{-11}=p,\ x_{-10}=m,\ x_{-9}=l,\ x_{-8}=k,\ x_{-7}=h,\ x_{-6}=g,\ x_{-5}=f,\ x_{-4}=e,\ x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b,\ x_{-0}=a$.*where* $x_{-8}=k,\ x_{-7}=h,\ x_{-6}=g,\ x_{-5}=f,\ x_{-4}=e,\ x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b,\ x_{-0}=a$.

Theorem 8 *Eq.(6) has a unique equilibrium point which is the number zero.*

Example 5. Assume that $x_{-11}=0.5,\ x_{-10}=2,\ x_{-9}=8,\ x_{-8}=0.7,\ x_{-7}=3,\ x_{-6}=5,\ x_{-5}=1/3,\ x_{-4}=3,\ x_{-3}=2/5,\ x_{-2}=0.2,\ x_{-1}=6,\ x_0=0.1$ see Fig. 5

Example 6. See Fig. 6 since $x_{-11}=1.5,\ x_{-10}=2,\ x_{-9}=-1.8,\ x_{-8}=7,\ x_{-7}=3,\ x_{-6}=5,\ x_{-5}=-5,\ x_{-4}=3,\ x_{-3}=7,\ x_{-2}=2,\ x_{-1}=6,\ x_0=9$.

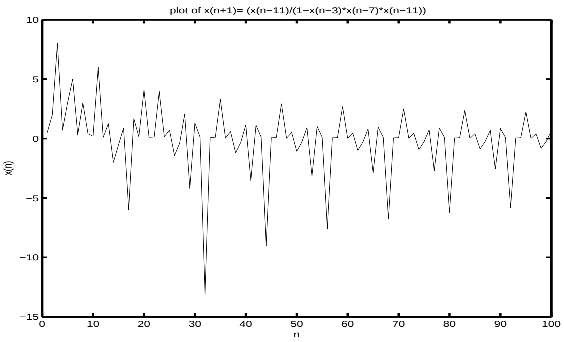

Figure 5: Example 5

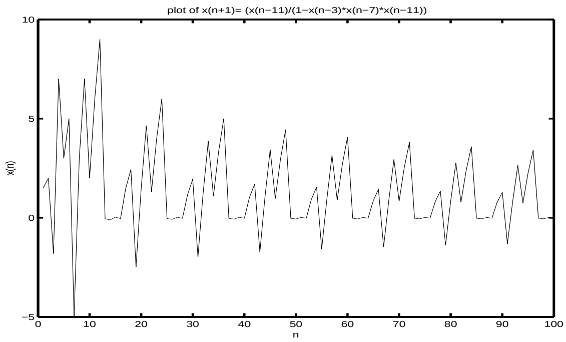

Figure 6: Example 6

5 Fourth Equation $x_{n+1}=\dfrac{x_{n-11}}{-1-x_{n-3}x_{n-7}x_{n-11}}$

Here we obtain a form of the solutions of the equation

$$x_{n+1}=\frac{x_{n-11}}{-1-x_{n-3}x_{n-7}x_{n-11}},\quad n=0,1,... \tag{7}$$

where the initial values are arbitrary non zero real numbers with $x_{-11}x_{-7}x_{-3}\neq-1,\ x_{-10}x_{-6}x_{-2}\neq-1,\ x_{-9}x_{-5}x_{-1}\neq-1,\ x_{-8}x_{-4}x_0\neq-1$.

Theorem 9 *Let $\{x_n\}_{n=-11}^{\infty}$ be a solution of Eq.(7). Then every solution of Eq.(7) is periodic with period 24 and for $n=0,1,...$*

$$
\begin{aligned}
x_{24n-11} &= p, \quad x_{24n-10}=m, \quad x_{24n-9}=l, \quad x_{24n-8}=k,\\
x_{24n-7} &= h, \quad x_{24n-6}=g, \quad x_{24n-5}=f, \quad x_{24n-4}=e,\\
x_{24n-3} &= d, \quad x_{24n-2}=c, \quad x_{24n-1}=b, \quad x_{24n}=a,\\
x_{24n+1} &= \frac{p}{-1-phd}, \quad x_{24n+2}=\frac{m}{-1-mgc}, \quad x_{24n+3}=\frac{l}{-1-lfb},\\
x_{24n+4} &= \frac{k}{-1-kea}, \quad x_{24n+5}=h\,(-1-phd)\,, \quad x_{24n+6}=g\,(-1-mgc)\,,\\
x_{24n+7} &= f\,(-1-lfb)\,, \quad x_{24n+8}=e\,(-1-kea)\,, \quad x_{24n+9}=\frac{d}{-1-phd},\\
x_{24n+10} &= \frac{c}{-1-mgc}, \quad x_{24n+11}=\frac{b}{-1-lfb}, \quad x_{24n+12}=\frac{a}{-1-kea},
\end{aligned}
$$

where $x_{-11} = p,\ x_{-10} = m,\ x_{-9} = l,\ x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a$.*where* $x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_{-0} = a$.

Theorem 10 *Eq.(7) has two equilibrium points which are* $0,\ \sqrt[3]{-2}$.

Theorem 11 *Eq.(7) has a periodic solutions of period twelve iff* $phd = mgc = lfb = kea = -2$ *and will be take the form* $\{p, m, l, k, h, g, f, e, d, c, b, a, p, m, l, k, h, g, f, e, d, c, b, a, ...\}$.

Example 7. Consider $x_{-11} = -1.5,\ x_{-10} = 2,\ x_{-9} = 1.8, x_{-8} = -0.7,\ x_{-7} = -3,\ x_{-6} = 1.5,\ x_{-5} = 0.5,\ x_{-4} = 1.3,\ x_{-3} = 0.7,\ x_{-2} = 1.2,\ x_{-1} = 0.6,\ x_0 = 0.9$ see Fig. 7
Example 8. Fig. 8. shows the solutions when $x_{-11} = 0.5,\ x_{-10} = -2,\ x_{-9} = 4, x_{-8} = 7,\ x_{-7} = 3,\ x_{-6} = 1/5, x_{-5} = -3,\ x_{-4} = 3,\ x_{-3} = -4/3,\ x_{-2} = 5,\ x_{-1} = 1/6,\ x_0 = -2/21$.

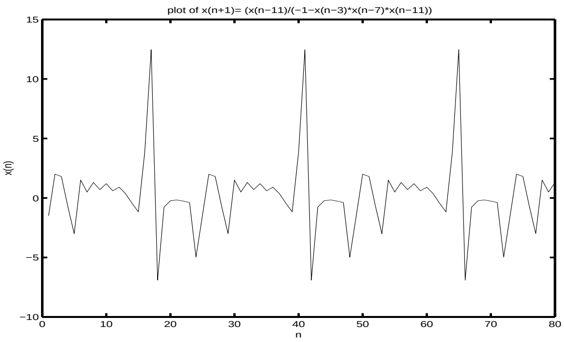

Figure 7: Example 7

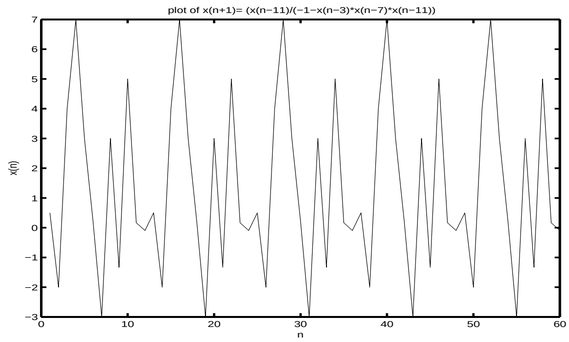

Figure 8: Example 8

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