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A Second-order Hybrid Finite Difference Scheme for a System of Coupled Singularly Perturbed Initial Value Problems

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Abstract: A system of coupled singularly perturbed initial value problems with a small parameter is considered. The solution to the system have boundary layers. The structure of these layers is analyzed, and this leads to the construction of a piecewise-uniform Shishkin mesh. On this mesh a hybrid finite difference scheme is proved to be almost second-order accurate, uniformly in the small parameter. Numerical results supporting the theory are presented.

Keywords: singular perturbation; hybrid finite difference scheme; Shishkin mesh; uniform convergence

1 Introduction

Singular perturbation problems arise in several branches of engineering and applied mathematics, including fluid dynamics, quantum mechanics, elasticity, chemical reactor, gas porous electrodes theory, etc. The presence of small parameter in these problems prevents us from obtaining satisfactory numerical solutions. It is well-known fact that the solutions of singular perturbation problems have a multi-scale character. That is, there is thin layer where the solution varies very rapidly, while away from the layer the solution behaves regularly and varies slowly. For the past two decades an extensive research has been made on numerical methods for the singularly perturbed differential equations, see [1,5-8] and the references therein. Robust numerical techniques have been developed for singularly perturbed problems, but for system of equations only few results are reported in the literature.

In this paper, we focus on a system of singularly perturbed initial value problem

$$\varepsilon u_1'(x) + f_1(x, u_1, u_2) = 0, \quad 0 < x \le 1, \tag{1.1}$$
$$\varepsilon u_2'(x) + f_2(x, u_1, u_2) = 0, \quad 0 < x \le 1, \tag{1.2}$$
$$u_1(0) = A, \quad u_2(0) = B, \tag{1.3}$$

where the parameter $\varepsilon \in (0, 1]$ is a small positive constant. We also assume that $f_1(x, u_1, u_2)$ and $f_2(x, u_1, u_2)$ are sufficiently smooth functions satisfying certain regularity conditions. These conditions will be specified whenever necessarily. Furthermore, we assume

$$0 < \beta \le \frac{\partial f_k}{\partial u_k} < +\infty, \quad -\infty < \frac{\partial f_k}{\partial u_{3-k}} \le 0, \quad k = 1, 2, \quad \text{in } [0, 1] \times R^2, \tag{1.4}$$
$$\min\{\frac{\partial f_1}{\partial u_1} + \frac{\partial f_1}{\partial u_2}, \frac{\partial f_2}{\partial u_2} + \frac{\partial f_2}{\partial u_1}\} \ge \alpha > 0, \quad \text{in } [0, 1] \times R^2. \tag{1.5}$$

The solution $u = (u_1, u_2)$ of problem (1.1)-(1.3) has boundary layers of width $O(\varepsilon|\ln\varepsilon|)$ at $x = 0$.

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A few results for systems of singularly perturbed second-order linear differential equations are available in the literature: Bellew and O'Riordan [3], Cen [5], Linß and Madden [10], Madden and Stynes [11], Shishkin [12] and Xi al et [13]. In [2] Amiraliyev consider a singularly perturbed system with the leading term of one equation multiplied by a small parameter.

This present study is devoted to a hybrid finite difference method for the coupled initial value equations (1.1)-(1.3) on a Shishkin mesh. We first prove bounds for $\mathbf{u}(x) = (u_1(x), u_2(x))$ and its derivatives. These bounds enable us to construct a piecewise-uniform Shishkin mesh on which we can prove that the scheme is almost second-order convergent, in the discrete maximum norm, independently of singular perturbation parameter ε.

An outline of the paper is as follows. In the next section we state some important properties of the exact solution. In section 3 we describe the hybrid finite difference scheme and introduce a Shishkin mesh. In section 4 we analyze the convergence properties of the scheme. Finally the numerical results are presented in section 5.

Notation 1 *Throughout the paper, C will denote a generic positive constant (possibly subscripted) that is independent of ε and of the mesh. Note that C is not necessarily the same at each occurrence. To simplify the notation we set $g_i = g(x_i)$ and $g_{i-1/2} = g((x_{i-1} + x_i)/2)$ for any function $g(x)$, while g_i^N denotes an approximation of $g(x)$ at x_i. We also define*

$$\mathbf{y}(x) = (y_1(x), y_2(x)), \quad \|\mathbf{y}(x)\| = \max\{|y_1(x)|, |y_2(x)|\}.$$

Proposition 1 *Throughout the paper we shall assume that $\varepsilon \leq CN^{-1}$ as is generally the case in practice, because otherwise we can solve the problem in the classical way.*

2 Some analytical results

In this section, we first establish a maximum principle for the following problem. Then, using this principle, a stability result for the same problem is derived. Further, we need to know the asymptotic behavior of the exact solution for constructing layer-adapted meshes correctly.

We rewrite the nonlinear system (1.1)-(1.3) in the form

$$L_1\mathbf{u} \equiv \varepsilon u_1'(x) + a_1(x)u_1(x) + b_1(x)u_2(x) = F_1(x), \tag{2.1}$$

$$L_2\mathbf{u} \equiv \varepsilon u_2'(x) + a_2(x)u_2(x) + b_2(x)u_1(x) = F_2(x), \tag{2.2}$$

$$u_1(0) = A, \quad u_2(0) = B, \tag{2.3}$$

where

$$a_k(x) = \frac{\partial f_k}{\partial u_k}(x, \xi_k(x), \eta_k(x)), \quad b_k(x) = \frac{\partial f_k}{\partial u_{3-k}}(x, \xi_k(x), \eta_k(x)), \quad F_k = -f_k(x, 0, 0),$$
$$\xi_k(x) = \rho_k u_k(x), \ \ 0 < \rho_k < 1, \ \ \eta_k = \lambda_k u_k(x), \quad 0 < \lambda_k < 1, \quad k = 1, 2.$$

Lemma 1 *(Maximum principle) Assume that $L_1\mathbf{y} \geq 0, L_2\mathbf{y} \geq 0$ for $x \in (0, 1]$ and $y_1(0) \geq 0, y_2(0) \geq 0$, then $\mathbf{y}(x) \geq \mathbf{0}$ for all $x \in [0, 1]$.*

Proof. Let $y_1(p) = \min\limits_{x\in[0,1]} y_1(x)$ and $y_2(q) = \min\limits_{x\in[0,1]} y_2(x)$. Assume without loss of generality that $y_1(p) \leq y_2(q)$. If the lemma is not true. Then $y_1(p) < 0$. Note that $p \neq 0$ and $y_1'(p) = 0$.

$$\begin{aligned} L_1\mathbf{y}(p) &= \varepsilon y_1'(p) + a_1(p)y_1(p) + b_1(p)y_2(p) \\ &= (a_1(p) + b_1(p))y_1(p) + b_1(p)(y_2(p) - y_1(p)) < 0, \end{aligned}$$

which contradicts the hypotheses of the lemma.

An immediate consequence is the following stability result. ■

Lemma 2 *(Stability result) If $\mathbf{y}(x)$ is a smooth vector function, then*

$$\|\mathbf{y}(x)\| \le C\max\{|y_1(0)|, |y_2(0)|, \max_{x\in[0,1]}|L_1\mathbf{y}|, \max_{x\in[0,1]}|L_2\mathbf{y}|\} \quad \text{for } x \in [0,1].$$

Now we give bounds on the derivatives of the exact solution $\mathbf{y}(x)$ for problem (1.1)-(1.3). These bounds will be used later in the analysis of the uniform convergence of the finite difference approximations defined in section 3.

Lemma 3 *Suppose the conditions (1.4) and (1.5) are satisfied. Then the solution $\mathbf{u}(x)$ of problem (1.1)-(1.3) satisfies*

$$\|\mathbf{u}^{(k)}(0)\| \le C\varepsilon^{-k}, \quad k = 1, 2, 3.$$

Proof. From the equations (1.1) and (1.2) it follows that

$$|u_1^{'}(0)| \le \varepsilon^{-1}|f_1(0, A, B)| \le C\varepsilon^{-1},$$
$$|u_2^{'}(0)| \le \varepsilon^{-1}|f_2(0, A, B)| \le C\varepsilon^{-1}.$$

Hence, $\|\mathbf{u}^{'}(0)\| \le C\varepsilon^{-1}$, so the inequality is proved with $k = 1$. If $k > 1$, the result is obtained by induction and repeated differentiations of equations (1.1) and (1.2). ■

Lemma 4 *Suppose the conditions (1.4) and (1.5) are satisfied. Then the solution $\mathbf{u}(x)$ of problem (1.1)-(1.3) satisfies*

$$\|\mathbf{u}^{(k)}(x)\| \le C(1 + \varepsilon^{-k}e^{-\alpha x/\varepsilon}), \quad 0 \le x \le 1, \ \ k = 0, 1, 2, 3.$$

Proof. The proof is by induction. From Lemma 2 the inequality holds for $k = 0$. Assume that

$$\|\mathbf{u}^{(m)}(x)\| \le C(1 + \varepsilon^{-m}e^{-\alpha x/\varepsilon}) \quad \text{for all} \quad m \le k.$$

Differentiating both sides of (2.1) and (2.2) k times, respectively. Setting $\mathbf{z}(x) = \mathbf{u}^{(k)}(x)$, we have

$$\varepsilon z_1^{'}(x) + a_1(x)z_1(x) + b_1(x)z_2(x) = g_1(x),$$
$$\varepsilon z_2^{'}(x) + a_2(x)z_2(x) + b_2(x)z_1(x) = g_2(x),$$

where $\mathbf{g}(x)$ depends on $\mathbf{u}(x), \mathbf{a}(x), \mathbf{b}(x), \mathbf{f}(x)$ and their derivatives of order up to and including $k - 1$. Hence we can easily obtain

$$\|\mathbf{g}(x)\| \le C(1 + \varepsilon^{-k+1}e^{-\alpha x/\varepsilon}) \ \text{ for } \ x \in [0,1].$$

From Lemma 3, we have

$$\|\mathbf{z}(0)\| \le C\varepsilon^{-k}.$$

Consider the barrier function $\phi(x) = (\phi_1(x), \phi_2(x))$, where

$$\phi_i(x) = C(1 + \varepsilon^{-k}e^{-\alpha x/\varepsilon}), \qquad i = 1, 2.$$

Applying the maximum principle we can get the desired results. ■

3 Mesh and scheme

We shall consider a hybrid difference scheme on a piecewise-uniform Shishkin mesh. Let N, our discretisation parameter, be an even positive integer. Let σ denote a mesh transition parameter defined by

$$\sigma = \min\{\frac{1}{2}, \ \frac{2\varepsilon}{\alpha}\ln N\}.$$

Divide each of the subintervals $[0, \sigma]$ and $[\sigma, 1]$ into $N/2$ equidistant subintervals. For our analysis we assume that $\sigma = 2\alpha^{-1}\varepsilon \ln N$, since otherwise N^{-1} is exponentially small compared with ε.

The mesh width of each subinterval in $[0, \sigma]$ is h. We use the notation H for the width in $[\sigma, 1]$. These mesh widths satisfy

$$h = \frac{4}{\alpha}\varepsilon N^{-1} \ln N, \qquad N^{-1} \leq H \leq 2N^{-1}.$$

On the piecewise-uniform Shishkin mesh we propose the following hybrid finite difference scheme for approximating problem (1.1)-(1.3):

$$T_1^N \mathbf{U}_i = 0, \quad T_2^N \mathbf{U}_i = 0 \quad \text{for } i = 1, 2, \cdots, N, \tag{3.1}$$

$$U_{1,0} = A, \quad U_{2,0} = B, \tag{3.2}$$

where

$$T_k^N \mathbf{U}_i = \begin{cases} \varepsilon \frac{U_{k,i} - U_{k,i-1}}{h_i} + f_k(x_{i-1/2}, \frac{U_{1,i-1}+U_{1,i}}{2}, \frac{U_{2,i-1}+U_{2,i}}{2}), & i = 1, 2, \cdots, N/2, \\ \varepsilon \frac{U_{k,i} - U_{k,i-1}}{h_i} + f_k(x_i, U_{1,i}, U_{2,i}), & i = N/2+1, \cdots, N, \end{cases}$$

for $k = 1, 2$.

4 Analysis of the method

To investigate the convergence of the method, note that the error functions $\mathbf{z} = \mathbf{U} - \mathbf{u}$ are the solutions of the discrete problem

$$\begin{aligned} &\varepsilon \frac{z_{k,i} - z_{k,i-1}}{h_i} + f_k(x_{i-1/2}, \frac{U_{1,i-1} + U_{1,i}}{2}, \frac{U_{2,i-1} + U_{2,i}}{2}) \\ &- f_k(x_{i-1/2}, u_{1,i-1/2}, u_{2,i-1/2}) = \varepsilon(u'_{k,i-1/2} - \frac{u_{k,i} - u_{k,i-1}}{h_i}), \quad 1 \leq i \leq N/2, \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\varepsilon \frac{z_{k,i} - z_{k,i-1}}{h_i} + f_k(x_i, U_{1,i}, U_{2,i}) - f_k(x_i, u_{1,i}, u_{2,i}) \\ &= \varepsilon(u'_{k,i} - \frac{u_{k,i} - u_{k,i-1}}{h_i}), \qquad N/2 < i \leq N, \quad k = 1, 2, \end{aligned} \tag{4.2}$$

$$z_{1,0} = z_{2,0} = 0. \tag{4.3}$$

For $1 \leq i \leq N/2$, we use Taylor expansion for f_k about $x_{i-1/2}$ to obtain

$$\varepsilon \frac{z_{k,i} - z_{k,i-1}}{h_i} + a_{k,i} \frac{z_{k,i-1} + z_{k,i}}{2} + b_{k,i} \frac{z_{3-k,i-1} + z_{3-k,i}}{2} = R_{k,i}, \tag{4.4}$$

where

$$\begin{aligned} &a_{k,i} = \frac{\partial}{\partial u_k} f_k(x_{i-1/2}, \xi_{k,i}, \eta_{k,i}), \qquad b_{k,i} = \frac{\partial}{\partial u_{3-k}} f_k(x_{i-1/2}, \xi_{k,i}, \eta_{k,i}), \\ &R_{k,i} = \varepsilon(u'_{k,i-1/2} - \frac{u_{k,i} - u_{k,i-1}}{h_i}) + a_{k,i}(u_{k,i-1/2} - \frac{u_{k,i-1} + u_{k,i}}{2}) \\ &+ b_{k,i}(u_{3-k,i-1/2} - \frac{u_{3-k,i-1} + u_{3-k,i}}{2}), \end{aligned} \tag{4.5}$$

$k = 1, 2,$ $\xi_{k,i}, \eta_{k,i}$-intermediate values.

For $N/2 < i \leq N$, we also use Taylor expansion for f_k about x_i to obtain

$$\varepsilon \frac{z_{k,i} - z_{k,i-1}}{h_i} + a_{k,i} z_{k,i} + b_{k,i} z_{3-k,i} = R_{k,i}, \tag{4.6}$$

where

$$\begin{aligned} &a_{k,i} = \frac{\partial}{\partial u_k} f_k(x_i, \xi_{k,i}, \eta_{k,i}), \qquad b_{k,i} = \frac{\partial}{\partial u_{3-k}} f_k(x_i, \xi_{k,i}, \eta_{k,i}), \\ &R_{k,i} = \varepsilon(u'_{k,i} - \frac{u_{k,i} - u_{k,i-1}}{h_i}), \end{aligned} \tag{4.7}$$

$k = 1, 2,$ $\xi_{k,i}, \eta_{k,i}$-intermediate values.

Lemma 5 *The error functions* $z_k,\ k=1,2$, *satisfy the following inequalities*

$$\|z_k\|_\infty \le C(\|R_1\|_\infty + \|R_2\|_\infty),$$

where $\|\cdot\|_\infty$ *denote the discrete maximum norm:*

$$\|w\|_\infty \equiv \max_{0\le i\le N} |w_i|.$$

Proof. Applying the discrete maximum principle for the difference operator

$$L_1^N z_{1,i} = \begin{cases} \varepsilon\frac{z_{1,i}-z_{1,i-1}}{h_i} + a_{1,i}\frac{z_{1,i-1}+z_{1,i}}{2}, & 1\le i\le N/2,\\ \varepsilon\frac{z_{1,i}-z_{1,i-1}}{h_i} + a_{1,i}z_{1,i}, & N/2 < i \le N \end{cases}$$

to equation (4.4) and (4.6) with $k=1$, we can obtain

$$\|z_1\|_\infty \le \max_{1\le i\le N} |\frac{1}{a_{1,i}}(R_{1,i} - b_{2,i}\frac{z_{2,i-1}+z_{2,i}}{2})| \le \alpha^{-1}\|R_1\|_\infty + \|\frac{b_1}{a_1}\|_\infty\|z_2\|_\infty. \tag{4.8}$$

Using the discrete maximum principle for the difference operator

$$L_2^N z_{2,i} = \begin{cases} \varepsilon\frac{z_{2,i}-z_{2,i-1}}{h_i} + a_{2,i}\frac{z_{2,i-1}+z_{2,i}}{2}, & 1\le i\le N/2,\\ \varepsilon\frac{z_{2,i}-z_{2,i-1}}{h_i} + a_{2,i}z_{2,i}, & N/2 < i \le N \end{cases}$$

to Eq.(4.4) and (4.6) with $k=2$, we also can obtain

$$\|z_2\|_\infty \le \max_{1\le i\le N} |\frac{1}{a_{2,i}}(R_{2,i} - b_{2,i}\frac{z_{1,i-1}+z_{1,i}}{2})| \le \alpha^{-1}\|R_2\|_\infty + \|\frac{b_2}{a_2}\|_\infty\|z_1\|_\infty. \tag{4.9}$$

From Eq.(4.8) and (4.9) we can get the following stability bounds

$$\|z_k\| \le C(\|R_1\|_\infty + \|R_2\|_\infty), \quad k=1,2,$$

where we have used the assumption (1.4) and (1.5). ■

Next lemma gives the truncation error estimate.

Lemma 6 *The truncation errors* R_1 *and* R_2 *of the difference scheme satisfy*

$$\|R_1\|_\infty \le CN^{-2}\ln^2 N, \quad \|R_2\|_\infty \le CN^{-2}\ln^2 N.$$

Proof. From explicit expression (4.5) for $R_{k,i}$, we use Taylor expansion for u_k and $u_k^{'}$ about x_i to obtain

$$\begin{aligned} |R_{k,i}| &\le \varepsilon|u^{'}_{k,i-1/2} - \frac{u_{k,i}-u_{k,i-1}}{h_i}| + |a_{k,i}(u_{k,i-1/2} - \frac{u_{k,i-1}+u_{k,i}}{2})| \\ &\le \frac{3\varepsilon}{2}\int_{x_{i-1}}^{x_i} |u_k^{'''}(x)|(x-x_{i-1})\mathrm{d}x + C\int_{x_{i-1}}^{x_i} |u_k^{''}(x)|(x-x_{i-1})\mathrm{d}x \\ &\le C\int_{x_{i-1}}^{x_i}(1+\varepsilon^{-2}e^{-\alpha x/\varepsilon})(x-x_{i-1})\mathrm{d}x \end{aligned} \tag{4.10}$$

for $1\le i\le N/2,\ k=1,2$, where we have used Lemma 4. To bound the right-hand side of (4.10) we use the inequality

$$\int_c^d y(x)(x-c)^{(p-1)}\mathrm{d}x \le \frac{1}{p}\{\int_c^d y(x)^{1/p}\mathrm{d}x\}^p$$

which holds true for any positive monotonically decreasing function $y(x)$ on $[c,d]$ and for arbitrary $p\in N^+$; see [4]. Then we have

$$\begin{aligned} |R_{k,i}| &\le C\{\int_{x_{i-1}}^{x_i}(1+\varepsilon^{-1}e^{-\alpha x/(2\varepsilon)})\mathrm{d}x\}^2 = C\{h_i - \frac{2}{\alpha}e^{-\alpha x/(2\varepsilon)}|_{x_{i-1}}^{x_i}\}^2 \\ &= C\{h_i - \frac{2}{\alpha}e^{-\alpha x_i/(2\varepsilon)}(1-e^{\alpha h_i/(2\varepsilon)})\}^2 \le CN^{-2}\ln^2 N \end{aligned} \tag{4.11}$$

for $1 \le i \le N/2,\ k = 1, 2$.

From explicit expression (4.7) for $R_{k,i}$, we also use Taylor expansion for u_k about x_i to obtain

$$\begin{aligned}
|R_{k,i}| &\le \varepsilon|u'_{k,i} - \frac{u_{k,i} - u_{k,i-1}}{h_i}| \le \frac{3\varepsilon}{2}\int_{x_{i-1}}^{x_i} |u''_k(x)|\mathrm{d}x \\
&\le C\int_{x_{i-1}}^{x_i} (\varepsilon + \varepsilon^{-1}e^{-\alpha x/\varepsilon})\mathrm{d}x \le C(\varepsilon h_i - \frac{1}{\alpha}e^{-\alpha x/\varepsilon}|_{x_{i-1}}^{x_i}) \\
&= C[\varepsilon h_i - \frac{1}{\alpha}(e^{-\alpha x_i/\varepsilon} - e^{-\alpha x_{i-1}/\varepsilon})] \\
&\le C(\varepsilon N^{-1} + \frac{1}{\alpha}e^{-\alpha x_{N/2}/\varepsilon}) \le CN^{-2} \qquad (4.12)
\end{aligned}$$

for $N/2 < i \le N,\ k = 1, 2$, where we have used Assumption 1.

Combining (4.10)-(4.12) we complete the proof. ■

From the two previous lemmas we immediately obtain the main results.

Theorem 1 *Let* $\mathbf{u} = (u_1, u_2)$ *be the solution of problem (1.1)-(1.3) and* $\mathbf{U} = (U_1, U_2)$ *be the solution of finite difference scheme (3.1)-(3.2) on the Shishkin mesh. Then we have the following error estimate*

$$|u_k(x_i) - U_{k,i}| \le CN^{-2}\ln^2 N \quad \text{for} \quad i = 0, 1, \cdots, N,\ \ k = 1, 2.$$

5 Numerical experiments

In this section we verify experimentally the theoretical results obtained in the preceding section. Errors and convergence rates for the hybrid finite difference scheme are presented for the following test problem.

Example 1 *Consider the problem*

$$\begin{aligned}
&\varepsilon u'_1(x) + 2u_1(x) - \frac{1}{2}e^{-u_1(x)} - u_2 = x^2 - 1, \quad 0 < x \le 1, \\
&\varepsilon u'_2(x) + 2u_2 - \cos u_1(x) = e^x, \quad 0 < x \le 1, \\
&u_1(0) = 0, \quad u_2(0) = 0.
\end{aligned}$$

For our experiments we take $\varepsilon = 10^{-8}$ *and* 10^{-6}*, respectively, which are sufficiently small choices to bring out the singularly perturbed nature of the problem. The numerical solution is obtained by using iterative process. The exact solution of the test problem is not available. Therefore, we use the double mesh principle to estimate the errors and compute the experiment rates of convergence in our computed solution. We measure the accuracy in the discrete maximum norm*

$$e^N = \max\{\|U^N_{1,i} - U^{2N}_{1,2i}\|_\infty, \|U^N_{2,i} - U^{2N}_{2,2i}\|_\infty\},$$

the convergence rate

$$r^N = \log_2(\frac{e^N}{e^{2N}}).$$

The numerical results are clear illustrations of the convergence estimate of Theorem 1. They indicate that the theoretical results are fairly sharp.

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Table 1: The hybrid finite difference scheme for Example

	$\varepsilon = 10^{-8}$		$\varepsilon = 10^{-6}$	
N	*error*	*rate*	*error*	*rate*
32	1.9263e-2	1.552	1.9263e-2	1.552
64	6.5700e-3	1.600	6.5700e-3	1.600
128	2.1675e-3	1.615	2.1675e-3	1.615
256	7.0764e-4	1.665	7.0764e-4	1.665
512	2.2319e-4	1.696	2.2319e-4	1.696
1024	6.8865e-5	-	6.8865e-5	-

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