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L_p- solution of Weighted Cauchy-type Problem of a Diffre-integral Functional Equation

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Abstract: The topic of fractional calculus, (integration and differentiation of fractional-order) is a one of the singular integral and integro-differential operators, is enjoying interest among mathematicians, physicists and engineers (see [1]-[2], [5]-[10] and [12]-[13] and the references therein). In this work, we are concerned with a nonlinear weighted Cauchy type problem (For the earlier work and application see, for example, [6] and [8]) of a diffre-integral functional equation of fractional order. We will prove some local and global existence theorems for this problem, also we will study the uniqueness and stability of its solution.

Keywords: fractional calculus; weighted Cauchy-type problem; stability

1 Introduction

In an earlier work the author (see [8]) study the weighted Cauchy-type problem:

$$\begin{cases} D^{\alpha}\, u(t) \;=\; f(t,u), \;\; t \;>\; 0, \\ \quad t^{1-\alpha}\, u(t)|_{t=0} \;=\; b. \end{cases}$$

where the function $f(t,u)$ is assumed to be continuous on $R^+ \times R$, $|f(t,u)| \leq t^{\mu}e^{-\sigma t}\psi(t)|u|^m$, $\mu \geq 0, m > 1,\ \sigma > 0, \psi(t)$ is a continuous function on R^+.
Here, we deal with the nonlinear functional weighted Cauchy-type problem:

$$\begin{cases} D^{\alpha}\, u(t) \;=\; f(t,u(\phi(t))), \\ \quad t^{1-\alpha}\, u(t)|_{t=0} \;=\; b. \end{cases} \tag{1}$$

We investigate the behavior of solutions for problem (1) with certain nonlinearities, using the equivalence of the fractional diffre-integral problem with the corresponding Volterra integral equation, we prove the existence of L_p-solution such that the function f satisfies the Caratheodory conditions and the growth condition

$$|\, f(t,u)\,| \;\leq\; a(t) \;+\; k\,|\,u\,|, \quad \text{for each } \; t \in (0,1), \;\; u \in R,$$

where $a(.) \in L_p(0,1)$ and $k \geq 0$ be a constant. Moreover, we will study the uniqueness and the stability of the solution.

The plan of our paper is as follows. In the next section, we prepare some material needed to prove our results. Section 3 is devoted to our main results on the existence of some local and global solution. Section 4 is devoted to the uniqueness of the solution. In the last section, we prove the stability of the solution.

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2 Preliminaries

Let $L_1(I)$ be the class of Lebesgue integrable functions on the interval $I = [a, b]$, where $0 \leq a < b < \infty$ and let $\Gamma(.)$ be the gamma function. Also let $L_p(I)$ be the space of the functions with integrable p^{th} power with the norm $\|u\|_p = \left(\int_0^1 \mid u(t) \mid^p dt\right)^{\frac{1}{p}}, 1 \leq p \leq \infty$. Recall that the operator T is compact if it is continuous and maps bounded sets into relatively compact ones. The set of all compact operators from the subspace $U \subset X$ into the Banach space X is denoted by $C(U, X)$. Moreover, we set $B_r = \{u \in L_p(I) : ||u||_p < r, r > 0\}$.

Definition 2.1 *The fractional integral of the function $f(.) \in L_1(I)$ of order $\beta \in R^+$ is defined by (see [11] - [14])*

$$I_a^\beta f(t) = \int_a^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)} f(s)\, ds,$$

where (see [11]) $I^\beta I^\gamma f(t) = I^{\beta+\gamma} f(t),\ \beta,\ \gamma > 0.$

Definition 2.2 *The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0, 1)$ is defined as (see [11] - [14])*

$$D_a^\alpha f(t) = \frac{d}{dt} I_a^{1-\alpha} f(t), \quad t \in [a, b].$$

Now, let us recall some results which will be needed in the sequel.

Theorem 2.1 ***(Rothe Fixed Point Theorem)*** *[3]*
Let U be an open and bounded subset of a Banach space E, let $T \in C(\bar{U}, E)$. Then T has a fixed point if the following condition holds

$$T(\partial U) \subseteq \bar{U}.$$

Theorem 2.2 ***(Nonlinear alternative of Laray-Schauder type)*** *[3]*
Let U be an open subset of a convex set D in a Banach space E. Assume $0 \in U$ and $T \in C(\bar{U}, E)$. Then either

(A1) T has a fixed point in $\bar{U}$, or

(A2) there exists $\gamma \in (0, 1)$ and $x \in \partial U$ such that $x = \gamma\, Tx$.

Theorem 2.3 ***(Kolmogorov compactness criterion)*** *[4]*
Let $\Omega \subseteq L^p\,(0, 1),\ 1 \leq p < \infty$. If

(i) Ω is bounded in $L^p\,(0, 1)$ and

(ii) $x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$, then Ω is relatively compact in $L^p\,(0, 1)$, where

$$x_h(t) = \frac{1}{h} \int_t^{t+h} x(s)\, ds.$$

3 Existence of solution

We begin this section by proving the equivalence of problem (1) with the corresponding Volterra integral equation:

$$u(t) = b\, t^{\alpha-1} + \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(\phi(s)))\, ds, \quad t \in (0, 1). \tag{2}$$

Indeed: Let $u(t)$ be a solution of (2), multiply both sides of (2) by $t^{1-\alpha}$, we get

$$t^{1-\alpha}\, u(t) = b + t^{1-\alpha}\, I^\alpha\, f(t, u(\phi(t))),$$

which gives

$$t^{1-\alpha}u(t)|_{t=0} \;=\; b.$$

Now, operating by $I^{1-\alpha}$ on both sides of (2), then

$$I^{1-\alpha}u(t) \;=\; b_1 \;+\; I\, f(t, u(\phi(t))).$$

Differentiating both sides we get

$$D^{\alpha}\, u(t) \;=\; f(t, u(\phi(t))).$$

Conversely, let $u(t)$ be a solution of (1), integrate both sides, then

$$I^{1-\alpha}u(t) \;-\; I^{1-\alpha}u(t)|_{t=0} \;=\; I\, f(t, u(\phi(t))).$$

Operating by I^{α} on both sides of the last equation, then

$$Iu(t) \;-\; I^{\alpha}\, C \;=\; I^{1+\alpha}\, f(t, u(\phi(t))),$$

differentiate both sides, then

$$u(t) \;-\; C_1\, t^{\alpha-1} \;\;=\; I^{\alpha}\, f(t, u(\phi(t))),$$

from the initial condition, we find that $C_1 = b$, then we obtain (2), i.e. Problem (1) and equation (2) are equivalent to each other.
Now define the operator T as

$$Tu(t) \;=\; b\, t^{\alpha-1} \;+\; \int_0^t \frac{(t \;-\; s)^{\alpha-1}}{\Gamma(\alpha)}\, f(s, u(\phi(s)))\, ds, \;\; t \in (0,1).$$

To solve equation (2) it is necessary to find a fixed point of the operator T.
Now, we present our main result by proving some local and global existence theorems for the diffre-integral weighted Cauchy-type problem (1) in L_p . To facilitate our discussion, let us first state the following assumptions:

(i) $f : (0,1) \times R \rightarrow R$ be a function with the following properties:

(1) for each $t \in (0,1)$, $f(t,.)$ is continuous,

(2) for each $u \in R$, $f(.,u)$ is measurable,

(3) for each $t \in (0,1), u \in R$, $f(t,u)$ satisfies the growth condition

$$|\, f(t,u)\,| \;\leq\; a(t) \;+\; k\,|\,u\,|,$$

where $a(.) \in L_p(0,1)$ and $k \;\geq\; 0$ be a constant.

(ii) $\phi : (0,1) \rightarrow (0,1)$ is nondecreasing and there is a constant $M > 0$ such that $\phi' \geq M$ a.e. on $(0,1)$.

Now, for the local existence of the solutions we have the following theorem:

Theorem 3.1 *Let the assumptions (i) and (ii) are satisfied.*

$$\textit{If} \qquad k \;<\; M\,\Gamma(1+\alpha) \;\; \textit{and} \;\; p \;<\; \frac{1}{1-\alpha}, \tag{3}$$

then the diffre-integral weighted Cauchy-type problem (1) has a solution $u \in B_r$, where

$$r \;\leq\; \frac{\frac{b}{p\,(\alpha-1)+1} \;+\; \frac{1}{\Gamma(1+\alpha)}\,\|\,a\,\|_p}{1 \;-\; \frac{k}{M\,\Gamma(1+\alpha)}}.$$

Proof: Let u be an arbitrary element in B_r. Then from the assumptions (i) - (ii), we have

$$\begin{aligned}
||Tu||_p &= \left\{\int_0^1 \mid Tu(t) \mid^p dt\right\}^{\frac{1}{p}} \\
&\leq \left\{\int_0^1 \mid b\, t^{\alpha-1} \mid^p dt\right\}^{\frac{1}{p}} + \left\{\int_0^1 \mid \int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s,u(\phi(s)))\, ds \mid^p dt\right\}^{\frac{1}{p}} \\
&\leq \left\{\frac{b^p\ t^{p(\alpha-1)+1}}{p\,(\alpha-1)+1}|_0^1\right\}^{\frac{1}{p}} + \left\{\int_0^1 \left\{\int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ |f(s,u(\phi(s)))|\, ds\right\}^p dt\right\}^{\frac{1}{p}} \\
&\leq \frac{b}{p\,(\alpha-1)+1} + \left\{\int_0^1 \left\{\int_0^t \frac{(t\ -\ s)^{\alpha-1}}{\Gamma(\alpha)}\ (|a(s)|\ +\ k\,|u(\phi(s))|)\, ds\right\}^p dt\right\}^{\frac{1}{p}} \\
&\leq \frac{b}{p\,(\alpha-1)+1} + \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)}\, dt \left\{\int_0^1 (\mid a(s) \mid\ +\ k \mid u(\phi(s)) \mid)^p\, ds\right\}^{\frac{1}{p}} \\
&\leq \frac{b}{p\,(\alpha-1)+1} + \frac{1}{\Gamma(1+\alpha)} \left(\| a \|_p + \frac{k}{M}\left\{\int_0^1 \mid u(\phi(s)) \mid^p \phi'(s)\, ds\right\}^{\frac{1}{p}}\right) \\
&= \frac{b}{p\,(\alpha-1)+1} + \frac{1}{\Gamma(1+\alpha)} \left(\| a \|_p + \frac{k}{M}\left\{\int_{\phi(0)}^{\phi(1)} \mid u(x) \mid^p dx\right\}^{\frac{1}{p}}\right) \\
&\leq \frac{b}{p\,(\alpha-1)+1} + \frac{1}{\Gamma(1+\alpha)} \left\{\| a \|_p + \frac{k}{M}\, \|u\|_p\right\}.
\end{aligned}$$

The last estimate shows that the operator T maps L_p into itself. Now, let $u \in \partial B_r$, that is, $||u||_p = r$, then the last inequality implies

$$||Tu||_p \leq \frac{b}{p\,(\alpha-1)+1} + \frac{1}{\Gamma(1+\alpha)} \left\{\| a \|_p + \frac{k}{M}\, r\right\}$$

Then $T(\partial B_r) \subset \bar{B}_r$ (closure of B_r) if

$$||Tu||_p \leq \frac{b}{p\,(\alpha-1)+1} + \frac{1}{\Gamma(1+\alpha)} \left\{\| a \|_p + \frac{k}{M}\, r\right\} \leq r,$$

which implies that

$$\frac{b}{p\,(\alpha-1)+1} + \frac{1}{\Gamma(1+\alpha)} \left\{\| a \|_p + \frac{k}{M}\, r\right\} \leq r.$$

Therefore

$$r \leq \frac{\frac{b}{p\,(\alpha-1)+1} + \frac{1}{\Gamma(1+\alpha)}\, \| a \|_p}{1 - \frac{k}{M\,\Gamma(1+\alpha)}}.$$

Using inequality (3) we deduce that $r > 0$. Moreover, we have

$$\begin{aligned}
||f||_p &= \left(\int_0^1 \mid f(s,u(\phi(s))) \mid^p ds\right)^{\frac{1}{p}} \\
&\leq \left(\int_0^1 (\mid a(s) \mid\ +\ k \mid u(\phi(s)) \mid)^p\ ds\right)^{\frac{1}{p}} \\
&\leq ||a||_p + \frac{k}{M}\, ||u||_p\,.
\end{aligned}$$

This estimation shows that f in $L_p(0,1)$.
Further, f is continuous in u (assumption 1) and I^α maps $L_p(0,1)$ continuously into itself, $I^\alpha f(t,u(\phi(t)))$ is continuous in u. Since u is an arbitrary element in B_r, T maps B_r continuously into $L_p(0,1)$.

Now, we will show that T is compact, to achieve this goal we will apply Theorem 2.3. So, let Ω be a bounded subset of B_r. Then $T(\Omega)$ is bounded in $L_p(0,1)$, i.e. condition (i) of Theorem 2.3 is satisfied. It remains to show that $(Tu)_h \to Tu$ in $L_p(0,1)$ as $h \to 0$, uniformly with respect to $Tu \in T\,\Omega$. We have the following estimation:

$$\begin{aligned}
||(Tu)_h \, - \, Tu||_p \;&=\; \left\{\int_0^1 |\,(Tu)_h(t) \, - \, (Tu)(t)\,|^p \, dt\right\}^{\frac{1}{p}} \\
&=\; \left\{\int_0^1 |\,\frac{1}{h}\int_t^{t+h} (Tu)(s)\, ds \, - \, (Tu)(t)\,|^p \, dt\right\}^{\frac{1}{p}} \\
&\leq\; \left\{\int_0^1 \left(\frac{1}{h}\int_t^{t+h} |\,(Tu)(s) \, - (Tu)(t)\,|^p \, ds\right) dt\right\}^{\frac{1}{p}} \\
&\leq\; \left\{\int_0^1 \frac{1}{h}\int_t^{t+h} |\,b\, s^{\alpha-1} \, - \, b\, t^{\alpha-1}\,|^p \, ds\, dt\right\}^{\frac{1}{p}} \\
&+\; \left\{\int_0^1 \frac{1}{h}\int_t^{t+h} |\,I^\alpha\, f(s,u(\phi(s))) \, - \, I^\alpha\, f(t,u(\phi(t)))\,|^p \, ds\, dt\right\}^{\frac{1}{p}}.
\end{aligned}$$

Since $f \in L_p(0,1)$ we get that $I^\alpha f(.) \in L_p(0,1)$. Moreover $t^{\alpha-1} \in L_p(0,1)$. So, we have (see [15])

$$\frac{1}{h}\int_t^{t+h} |\,b\, s^{\alpha-1} \, - \, b\, t^{\alpha-1}\,|^p \, ds \;\to\; 0$$

and

$$\frac{1}{h}\int_t^{t+h} |\,I^\alpha\, f(s,u(\phi(s))) \, - \, I^\alpha\, f(t,u(\phi(t)))\,|^p \, ds \;\to\; 0$$

for a.e. $t \in (0,1)$. Therefore, by Theorem 2.3, we have that $T(\Omega)$ is relatively compact, that is, T is a compact operator.
Therefore, Theorem 2.1 with $U = B_r$ and $E = L_p(0,1)$ implies that T has a fixed point. This complete the proof. ∎

Now for more global solution of the diffre-integral weighted Cauchy-type problem (1), consider the following assumption:

(iii) Assume that every solution $u(.) \in L_p(0,1)$ to the equation

$$u(t) \;=\; \gamma\left(b\, t^{\alpha-1} \;+\; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f(s,u(\phi(s)))\, ds\right) \;\text{ a.e. on }\; (0,1),\; 0 < \alpha < 1$$

satisfies $||u||_p \neq r$ (r is arbitrary but fixed).

Theorem 3.2 *Let the conditions (i) - (iii) be satisfied, then the diffre-integral weighted Cauchy-type problem (1) has at least one solution $u \in L_p(0,1)$.*

Proof: Let u be an arbitrary element in the open set $B_r = \{u : ||u||_p < r, r > 0\}$. Then from the assumptions (i) - (ii), we have

$$||Tu||_p \;\leq\; \frac{b}{p\,(\alpha-1)+1} \;+\; \frac{1}{\Gamma(1+\alpha)}\,\left\{\|\,a\,\|_p \;+\; \frac{k}{M}\,\|u\|_p\right\}.$$

The above inequality means that the operator T maps B_r into L_p. Moreover, we have

$$||f||_p \;\leq\; ||a||_p \;+\; \frac{k}{M}\,||u||_p.$$

This estimation shows that f in $L_p(0,1)$.
As a consequence of Theorem 3.1 we get that T maps B_r continuously into $L_p(0,1)$ and T is compact.
Set $U = B_r$ and $D = E = L_p(0,1)$, then in the view of assumption (iii) the condition $A2$ of Theorem 2.2 does not hold. Therefore, Theorem 2.2 implies that T has a fixed point. This complete the proof. ∎

4 Uniqueness of the solution

For the uniqueness of the solution we have the following theorem:

Theorem 4.1 *Let the assumptions of Theorem 3.1 be satisfied, but instead of assumption (i) consider the following conditions:*

$$| f(t,u) - f(t,v) | \leq L | u - v |$$

and

$$| f(t,0) | \leq a(t),$$

then the diffre-integral weighted Cauchy-type problem (1) has a unique solution.

Proof: Let $u_1(t)$ and $u_2(t)$ be any two solutions of equation (2), then

$$\begin{aligned} | u_2(t) - u_1(t) |^p &= \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{f(s,u_2(\phi(s))) - f(s,u_1(\phi(s)))\} \, ds \right|^p \\ &\leq L^p \left(\int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} | u_2(\phi(s)) - u_1(\phi(s)) | \, ds \right)^p . \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^1 | u_2(t) - u_1(t) |^p \, dt &\leq L^p \int_0^1 \left(\int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} | u_2(\phi(s)) - u_1(\phi(s)) | \, ds \right)^p dt, \\ || u_2 - u_1 ||_p &\leq L \left\{ \int_0^1 \left(\int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} | u_2(\phi(s)) - u_1(\phi(s)) | \, ds \right)^p dt \right\}^{\frac{1}{p}} \\ &\leq L \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} \, dt \left(\int_0^1 | u_2(\phi(s)) - u_1(\phi(s)) |^p \, ds \right)^{\frac{1}{p}} \\ &\leq \frac{L}{\Gamma(1+\alpha)} \left(\int_0^1 | u_2(\phi(s)) - u_1(\phi(s)) |^p \frac{\phi'(s)}{M} \, ds \right)^{\frac{1}{p}} \\ &\leq \frac{L}{\Gamma(1+\alpha)} \left(\frac{1}{M} \int_{\phi(0)}^{\phi(1)} | u_2(x) - u_1(x) |^p \, dx \right)^{\frac{1}{p}} \\ &\leq \frac{L}{M \, \Gamma(1+\alpha)} || u_2 - u_1 ||_p . \quad \blacksquare \end{aligned}$$

5 Stability

Now we study the stability of the diffre-integral weighted Cauchy-type problem (1).

Theorem 5.1 *Let the assumptions of Theorem 4.1 be satisfied, then the solution of the diffre-integral weighted Cauchy-type problem (1) is uniformly stable.*

Proof: Let $u(t)$ be a solution of

$$u(t) = b \, t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(\phi(s))) \, ds,$$

and let $\widetilde{u}(t)$ be a solution of the above equation such that $t^{1-\alpha}\, \widetilde{u}(t)|_{t=0} = \widetilde{b}$, then

$$\begin{aligned}
u(t) - \widetilde{u}(t) &= (b - \widetilde{b})\, t^{\alpha-1} \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{f(s, u(\phi(s))) - f(s, \widetilde{u}(\phi(s)))\}\, ds, \\
|\, u(t) - \widetilde{u}(t)\,|^p &= |\,(b - \widetilde{b})\, t^{\alpha-1} \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{f(s, u(\phi(s))) - f(s, \widetilde{u}(\phi(s)))\}\, ds\,|^p, \\
\left(\int_0^1 |u(t) - \widetilde{u}(t)|^p\, dt\right)^{\frac{1}{p}} &\leq \left\{\int_0^1 |\,(b - \widetilde{b})\, t^{\alpha-1}\,|^p\, dt\right\}^{\frac{1}{p}} \\
&+ \left\{\int_0^1 \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{f(s, u(\phi(s))) - f(s, \widetilde{u}(\phi(s)))\}\, ds\right|^p dt\right\}^{\frac{1}{p}} \\
\|\, u - \widetilde{u}\,\|_p &\leq \left\{\frac{|\, b - \widetilde{b}\,|^p\, t^{p(\alpha-1)+1}}{p\,(\alpha-1)+1}|_0^1\right\}^{\frac{1}{p}} \\
&+ \left\{\int_0^1 \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(\phi(s))) - f(s, \widetilde{u}(\phi(s)))|\, ds\right)^p dt\right\}^{\frac{1}{p}} \\
&\leq \frac{|\, b - \widetilde{b}\,|}{p\,(\alpha-1)+1} \\
&+ L\left\{\int_0^1 \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, |\, u(\phi(s)) - \widetilde{u}(\phi(s))\,|\, ds\right)^p dt\right\}^{\frac{1}{p}} \\
&\leq \frac{|\, b - \widetilde{b}\,|}{p\,(\alpha-1)+1} \\
&+ L\int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)}\, dt \left(\int_0^1 |\, u(\phi(s)) - \widetilde{u}(\phi(s))\,|^p\, ds\right)^{\frac{1}{p}} \\
&\leq \frac{|\, b - \widetilde{b}\,|}{p\,(\alpha-1)+1} \\
&+ \frac{L}{\Gamma(1+\alpha)} \left(\int_0^1 |\, u(\phi(s)) - \widetilde{u}(\phi(s))\,|^p\, \frac{\phi'(s)}{M}\, ds\right)^{\frac{1}{p}} \\
&= \frac{|\, b - \widetilde{b}\,|}{p\,(\alpha-1)+1} + \frac{L}{\Gamma(1+\alpha)} \left(\frac{1}{M}\int_{\phi(0)}^{\phi(1)} |u(x) - \widetilde{u}(x)|^p\, dx\right)^{\frac{1}{p}} \\
&\leq \frac{|\, b - \widetilde{b}\,|}{p\,(\alpha-1)+1} + \frac{L}{M\,\Gamma(1+\alpha)}\, ||\, u - \widetilde{u}\,||_p\,, \\
\left(1 - \frac{L}{M\,\Gamma(1+\alpha)}\right) \|\, u - \widetilde{u}\,\|_p &\leq \frac{|\, b - \widetilde{b}\,|}{p\,(\alpha-1)+1}, \\
\|\, u - \widetilde{u}\,\|_p &\leq \left(1 - \frac{L}{M\,\Gamma(1+\alpha)}\right)^{-1} \frac{|\, b - \widetilde{b}\,|}{p\,(\alpha-1)+1}.
\end{aligned}$$

Therefore, if $|\, b - \widetilde{b}\,| < \delta(\varepsilon)$, then $||\, u - \widetilde{u}\,||_p < \varepsilon$. Now from the equivalence we get that the solution of the diffre-integral weighted Cauchy-type problem (1) is uniformly stable. ■

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