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Reaction-Diffusion Systems and Stability Properties of Positive Solutions

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Abstract: This paper deals with the stability of positive solutions to the systems of the form

$$\begin{cases} -\triangle u_i(x) = f_i(u_1(x), u_2(x), ..., u_n(x)) & x \in \Omega, \\ Bu_i(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset R^N(N \geq 1)$ is a bounded domain with a smooth boundary $Bz(x) = \alpha h(x)z + (1-\alpha)\frac{\partial z}{\partial n}$ where $\alpha \in [0,1], h : \partial\Omega \to R^+$ with $h = 1$ when $\alpha = 1$, and $f_i : [0,+\infty)^n \to R$ are C^1 functions for $i = 1, ..., n$. In particular, we establish conditions for stability/instability when the system is cooperative and strictly coupled ($\frac{\partial f_i}{\partial u_j} \geq 0, i \neq j, \sum_{j=1,j\neq i}^{n}(\frac{\partial f_i}{\partial u_j})^2 > 0$) and strictly coupled competitive ($\frac{\partial f_i}{\partial u_j} \leq 0, i \neq j$). Then we apply our results to various examples.

Keywords:reaction-diffusion systems; stability properties; positive solutions
AMS subject classification: 35J60, 35B30, 35B40

1 Introduction

We discuss stability of positive solutions $u := (u_1, ..., u_n)$ ($u_i \geq 0$ and $\sum_{i=1}^{n} u_i^2 > 0$) of systems of the form

$$-\triangle u_i(x) = f_i(u_1(x), u_2(x), ..., u_n(x)) \qquad x \in \Omega, \tag{1}$$

$$Bu_i(x) = 0 \qquad x \in \partial\Omega, \tag{2}$$

for $i = 1, ..., n$, where Ω is a bounded region in $R^N; N \geq 1$ with smooth boundary

$$Bz(x) = \alpha h(x)z + (1-\alpha)\frac{\partial z}{\partial n}, \tag{3}$$

here B is a boundary operator and $(\frac{\partial}{\partial n})$ denotes the outward conormal derivative, $\alpha \in [0,1], h : \partial\Omega \to R^+$ with $h = 1$ when $\alpha = 1$, i.e; the boundary condition may be of Dirichlet($u = 0$ on $\partial\Omega$), Neumann ($\frac{\partial z}{\partial n} = 0$ on $\partial\Omega$) or mixed type (robin boundary condition (for $\alpha \neq 0, 1$)). For i=1,...,n, $f_i : [0,+\infty)^n \to R$ are C^1 functions satisfying either

$$\frac{\partial f_i}{\partial u_j} \geq 0, i \neq j, j = 1, ..., n, \sum_{j=1,j\neq i}^{n}(\frac{\partial f_i}{\partial u_j})^2 > 0 \tag{4}$$

(strictly coupled cooperative system) or

$$\frac{\partial f_i}{\partial u_j} \leq 0, i \neq j, j = 1, 2, n = 2 \tag{5}$$

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(strictly coupled competitive system).

One can think of a solution $u := (u_1, ..., u_n)$ of (1)-(2) as an equilibrium solution to the parabolic problem

$$\begin{cases} (v_i)_t - \triangle v_i = f_i(v_1, v_2, ..., v_n) & in\ (0,\infty)\times\Omega, \\ Bv_i(t,x) = 0 & on\ (0,\infty)\times\partial\Omega, \\ v_i(0,x) = k_i(x) & in\ \Omega, \end{cases} \tag{6}$$

for i=1,...,n. Hence, denoting by $k(x) := (k_1(x), ..., k_n(x))$ and the corresponding solution of (6) by $v(t,x) := (v_1(t,x), ..., v_n(t,x))$, we say a positive solution u of (1)-(2) is stable (in the maximum norm) if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|k - u\|_\infty < \delta$ implies $\|v(t,x) - u\|_\infty < \epsilon$ for all $t > 0$. We say u is unstable if it is not stable.

As a reminder, if $\tilde{u} = (\bar{u}_1, ..., \bar{u}_n)$ be any positive solution to system (1)-(2), then the linearized equation about $\bar{u}$ consists of

$$-\triangle(\phi_i(x)) - \sum_{j=1}^{n}\frac{\partial f_i}{\partial u_j}(\bar{u})\phi_j(x) = \mu\phi_i(x), \qquad x\in\Omega, \tag{7}$$

$$B\phi_i(x) = 0, \qquad x\in\partial\Omega, \tag{8}$$

for i=1,...,n, where Equation (7) obtained from the formal derivative of the operator $\triangle$.

It follows from proposition (3.1) of [6] that there is a unique eigenvalue μ_1 with strictly positive eigenfunction $\phi = (\phi_1, ..., \phi_n)$ of (7)-(8) (i.e., $\phi_i > 0$ in Ω for every i=1,...,n). By proof same the proof of Theorem 1.1 and 1.2 of [3] for cooperative systems, one can show that if $\mu_1 < 0$ then positive solution $\bar{u}$ of (1)-(2) is unstable, otherwise $\bar{u}$ is stable. Hence we by this fact establish our results.

A large number of works have been made extensively studying stability of solutions in the case when $n = 1$ (see [1],[2],[4],[5]). In [3] authors studied case $n > 1$ with only Dirichlet boundary condition. The purpose of this work is to extend their results to Eqs. (1)-(2), under certain conditions.

2 Main results

In this section, by analyzing linearized equation, we will prove the stability of positive solutions.

Theorem 1 *Let (4) hold, and assume that for i=1,...,n,*

$$\sum_{j=1}^{n}\bar{u}_j(x)\frac{\partial f_j}{\partial u_i}(\bar{u}) > (<) f_i(\bar{u}), \tag{9}$$

then any positive solution of Equations (1)-(2) is linearly unstable(stable).

Proof. Let $\bar{u} := (\bar{u}_1, ..., \bar{u}_n)$ be any positive solution to (1)-(2). For each i=1,...,n, multiplying (7) by $-\bar{u}_i$, (1) by ϕ_i, add integrating by parts over Ω and add the two resulting expressions to get

$$\int_\Omega[\bar{u}_i(x)\triangle\phi_i - \phi_i(x)\triangle\bar{u}_i]dx + \int_\Omega\sum_{j=1}^{n}\bar{u}_i(x)\frac{\partial f_i}{\partial u_j}(\bar{u})\phi_j(x)dx - \int_\Omega f_i(\bar{u})\phi_i(x)dx$$
$$= -\mu_1\int_\Omega\bar{u}_i(x)\phi_i(x)dx. \tag{10}$$

But by Green identity

$$\int_\Omega[\bar{u}_i(x)\triangle\phi_i - \phi_i(x)\triangle\bar{u}_i]dx = \int_{\partial\Omega}[\bar{u}_i(x)\frac{\partial\phi_i}{\partial n} - \phi_i(x)\frac{\partial\bar{u}_i}{\partial n}]ds, \tag{11}$$

now observe that in (3) when $\alpha = 1$ then $B\bar{u}_i = \bar{u}_i = 0$ for all $s\in\partial\Omega$ and because ϕ_i are eigenfunctions, thus $\phi_i = 0$ for all $s\in\partial\Omega$. Hence

$$\int_{\partial\Omega}[\bar{u}_i(x)\frac{\partial\phi_i}{\partial n} - \phi_i(x)\frac{\partial\bar{u}_i}{\partial n}]ds = 0, \tag{12}$$

and when $\alpha \neq 1$, we obtain

$$\int_{\partial\Omega}[\bar{u}_i(x)\frac{\partial\phi_i}{\partial n} - \phi_i(x)\frac{\partial\bar{u}_i}{\partial n}]ds = \int_{\partial\Omega}\frac{\alpha h\phi_i(s)}{(1-\alpha)}(\bar{u}_i - \bar{u}_i)ds = 0. \tag{13}$$

Thus using (12) and (13) in (11), (10) becomes

$$\int_\Omega \sum_{j=1}^{n}\bar{u}_i(x)\frac{\partial f_i}{\partial u_j}(\bar{u})\phi_j(x)dx - \int_\Omega f_i(\bar{u})\phi_i(x)dx = -\mu_1\int_\Omega \bar{u}_i(x)\phi_i(x)dx, \tag{14}$$

now adding (14) over i=1,...,n, and then rearranging the terms, we have

$$\int_\Omega \sum_{i=1}^{n}\phi_i(x)((\sum_{j=1}^{n}u_j(x)\frac{\partial f_j}{\partial u_i}) - f_i(\bar{u}))dx = -\mu_1\int_\Omega\sum_{i=1}^{n}\bar{u}_i(x)\phi_i(x)dx. \tag{15}$$

By hypothesis, for each i=1,...,n, $\bar{u}_i > 0$ and $\phi_i > 0$ in Ω, thus if (9) holds, then $\mu_1 < 0(> 0)$ and it show that every positive solution of $(1) - (2)$ is linearly unstable(stable). ■

Theorem 2 *Let (4) hold, and assume that for i=1,...,n, $f_i(u_1, ..., u_n)/u_i$ with respect to u_i be strictly increasing, then any positive solution of Equations (1)-(2) is linearly unstable.*

Proof. Let $\bar{u} := (\bar{u}_1, ..., \bar{u}_n)$ be any positive solution to (1)-(2). The proof proceeds in the same way as for Theorem 1., instead of (15) we obtain

$$\int_\Omega\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\bar{u}_i(x)\frac{\partial f_i}{\partial u_j}(\bar{u})\phi_j(x)dx + \int_\Omega\sum_{i=1}^{n}\phi_i(x)[\bar{u}_i(x)\frac{\partial f_i}{\partial u_i}(\bar{u}) - f_i(\bar{u})]dx$$

$$= -\mu_1\int_\Omega\sum_{i=1}^{n}\bar{u}_i(x)\phi_i(x)dx. \tag{16}$$

On the other hand since $f_i(u_1, ..., u_n)/u_i$ with respect to u_i is strictly decreasing,

$$(\frac{f_i}{u_i})_{u_i}(\bar{u}_i) = \frac{\bar{u}_i(f_i)_{u_i}(\bar{u}_i) - f_i(\bar{u}_i)}{\bar{u}_i^2} > 0 \tag{17}$$

(here $(\frac{f_i}{u_i})_{u_i}$ denotes the partial derivative of $\frac{f_i}{u_i}$ with respect to u_i). Also $\bar{u}_i > 0$, $\phi_i > 0$ in Ω, and by (4), we have

$$-\mu_1\int_\Omega\sum_{i=1}^{n}\bar{u}_i(x)\phi_i(x)dx > 0.$$

So, $\mu_1 < 0$, hence any positive solution of Equations (1)-(2) is linearly unstable. ■

Theorem 3 *Let (5) hold, and assume that for i=1,2*

$$\sum_{j=1}^{2}(-1)^{i+j}\bar{u}_j(x)\frac{\partial f_j}{\partial u_i}(\bar{u}) > (<) f_i(\bar{u}), \tag{18}$$

then any positive solution of Equations (1)-(2) is linearly unstable(stable).

Proof. Let $\bar{u} := (\bar{u}_1, \bar{u}_2)$ be any positive solution to (1)-(2). Let $v_1 = u_1$ and $v_2 = -u_2$ then

$$-\triangle v_1(x) = f_1(u_1, u_2) = f_1(v_1, -v_2) = F_1(v_1, v_2) \qquad x \in \Omega, \tag{19}$$

$$-\triangle v_2(x) = -f_2(u_1, u_2) = -f_2(v_1, -v_2) = F_2(v_1, v_2) \qquad x \in \Omega, \tag{20}$$

notice that $\frac{\partial F_i}{\partial v_j} = -\frac{\partial f_i}{\partial u_j} > 0$ for $j \neq i$, thus Equation (19)-(20) is cooperative with respect to F_1 and F_2. Now if let $\bar{v}_1 = \bar{u}_1$ and $\bar{v}_2 = -\bar{u}_2$ then the linearized equation about $\bar{v} = (\bar{v_1}, \bar{v_2})$ consists of

$$-\triangle(\phi_1(x)) - \frac{\partial F_1}{\partial v_1}(\bar{v})\phi_1(x) - \frac{\partial F_1}{\partial v_2}(\bar{v})\phi_2(x) = \mu\phi_1(x) \qquad x \in \Omega, \tag{21}$$

$$-\triangle(\phi_2(x)) - \frac{\partial F_2}{\partial v_1}(\bar{v})\phi_1(x) - \frac{\partial F_2}{\partial v_2}(\bar{v})\phi_2(x) = \mu\phi_2(x) \qquad x \in \Omega, \tag{22}$$

$$B\phi_1(x) = 0 = B\phi_2(x) \qquad x \in \partial\Omega, \tag{23}$$

calculate $-(19)\phi_1 + (21)\bar{v}_1 + (20)\phi_2 - (22)\bar{v}_2$ and integrate over Ω and then simplify to get

$$\mu_1 \int_\Omega [\bar{v}_1\phi_1 - \bar{v}_2\phi_2]dx = \int_\Omega \phi_1(x)[\bar{v}_2(x)\frac{\partial F_2}{\partial v_1}(\bar{v}) - \bar{v}_1(x)\frac{\partial F_1}{\partial v_1}(\bar{v}) + F_1(\bar{v})]dx$$

$$+ \int_\Omega \phi_2(x)[-\bar{v}_1(x)\frac{\partial F_1}{\partial v_2}(\bar{v}) + \bar{v}_2(x)\frac{\partial F_2}{\partial v_2}(\bar{v}) - F_2(\bar{v})]dx. \tag{24}$$

Now by hypothesis, $\mu_1 < 0 (> 0)$ if terms on right-hand side be negative (positive), that is if (18) holds. Hence every positive solution of (1)-(2) is unstable(stable). ■

3 Examples

Consider following problem

$$\begin{cases} -\triangle u(x) = \lambda f(u(x), v(x)) & x \in \Omega, \\ Bu = 0 = Bv & x \in \partial\Omega, \end{cases} \tag{25}$$

where λ is a positive parameter.

Example 1. For semipositone system

$$\begin{cases} f(u,v) = u^p + uv^\alpha - \epsilon_1 \\ g(u,v) = v^q + vu^\beta - \epsilon_2 \end{cases}$$

where $p, q > 1, \alpha, \beta > 0, \epsilon_1, \epsilon_2 > 0$, clearly we have $\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} > 0$, and

$$\sum_{j=1}^{2} u_j(x)\frac{\partial f_j}{\partial u_1} = u\frac{\partial f}{\partial u} + v\frac{\partial g}{\partial u} = pu^p + uv^\alpha + \beta v^2 u^{\beta-1} > f(u,v)$$

and

$$\sum_{j=1}^{2} u_j(x)\frac{\partial f_j}{\partial u_2} = u\frac{\partial f}{\partial v} + v\frac{\partial g}{\partial v} = \alpha u^2 v^{\alpha-1} + qv^q + vu^\beta > g(u,v).$$

Thus (9) satisfied and all positive solutions of (25) are unstable for all $\lambda > 0$. Or one can infer unstable by Theorem 2, because

$$(\frac{f(u,v)}{u})_u = (u^{p-1} + v^\alpha - \frac{\epsilon_1}{u})_u = (p-1)u^{p-2} + \frac{\epsilon_1}{u^2} > 0,$$

and

$$(\frac{g(u,v)}{v})_v = (v^{q-1} + u^\beta - \frac{\epsilon_2}{v})_v = (q-1)v^{q-2} + \frac{\epsilon_2}{v^2} > 0.$$

Example 2. For positone system

$$\begin{cases} f(u,v) = (u+1)^p + (v+1)^q + 1 \\ g(u,v) = u(v+1)^{q-1} + 1 \end{cases}$$

where $0 \le p < 1, q < 1$, clearly we have $\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} > 0$, then

$$\sum_{j=1}^{2} u_j(x)\frac{\partial f_j}{\partial u_1} = u\frac{\partial f}{\partial u} + v\frac{\partial g}{\partial u} = pu(u+1)^{p-1} + v(v+1)^{q-1} < p(u+1)^p + (v+1)^q < f(u,v)$$

and

$$\sum_{j=1}^{2} u_j(x)\frac{\partial f_j}{\partial u_2} = u\frac{\partial f}{\partial v} + v\frac{\partial g}{\partial v} = qu(v+1)^{q-1} + (q-1)uv(v+1)^{q-2} < (2q-1)u(v+1)^{q-1} < g(u,v)$$

Thus (9) satisfied and all positive solutions of (25) are stable for all $\lambda > 0$.

Example 3. For semipositone system

$$\begin{cases} f(u,v) = u^p - uv^\alpha - \epsilon_1 \\ g(u,v) = v^q - vu^\beta - \epsilon_2 \end{cases}$$

where $p, q > 1, \alpha, \beta > 0, \epsilon_1, \epsilon_2 > 0$, clearly we have $\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} < 0$, then

$$\sum\nolimits_{j=1}^{2} (-1)^{1+j} u_j(x) \frac{\partial f_j}{\partial u_1} = u\frac{\partial f}{\partial u} - v\frac{\partial g}{\partial u} = pu^p - uv^\alpha + \beta v^2 u^{\beta-1} > f(u,v)$$

and

$$\sum\nolimits_{j=1}^{2} (-1)^{2+j} u_j(x) \frac{\partial f_j}{\partial u_2} = -u\frac{\partial f}{\partial v} + v\frac{\partial g}{\partial v} = \alpha u^2 v^{\alpha-1} + qv^q - vu^\beta > g(u,v)$$

Thus (18) satisfied and all positive solutions of (25) are unstable for all $\lambda > 0$.

Example 4. For semipositone system

$$\begin{cases} f(u,v) = u^p + \frac{1}{(u+v+1)^\alpha} + \epsilon_1 \\ g(u,v) = v^q + \frac{1}{(u+v+1)^\alpha} + \epsilon_2 \end{cases}$$

where $0 < p, q, \alpha < 1, \alpha < 1, \epsilon_1, \epsilon_2 > 0$, clearly we have $\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} < 0$, then

$$\sum\nolimits_{j=1}^{2} (-1)^{1+j} u_j(x) \frac{\partial f_j}{\partial u_1} = u\frac{\partial f}{\partial u} - v\frac{\partial g}{\partial u} = pu^p - \frac{\alpha}{(u+v+1)^{\alpha+1}}(u-v) < f(u,v)$$

and

$$\sum\nolimits_{j=1}^{2} (-1)^{2+j} u_j(x) \frac{\partial f_j}{\partial u_2} = -u\frac{\partial f}{\partial v} + v\frac{\partial g}{\partial v} = qv^q - \frac{\alpha}{(u+v+1)^{\alpha+1}}(v-u) < g(u,v)$$

Thus (18) satisfied and all positive solutions of (25) are stable for all $\lambda > 0$.

Example 5. Consider system

$$\begin{cases} -\triangle u_i(x) = \lambda f_i(u_1, ..., u_n) & x \in \Omega, \\ Bu_i = 0 & x \in \partial\Omega, \end{cases} \tag{26}$$

where $f_i = u_1^{\alpha_{i1}} u_2^{\alpha_{i2}} ... u_n^{\alpha_{in}}$, $\alpha_{ik} > 1$, for $i, k = 1, ..., n$, and $\lambda > 0$ is a parameter, clearly we have $\frac{\partial f_i}{\partial u_j} = \alpha_{ij} u_j^{\alpha_{ij}-1} \prod_{k=1,k\neq j}^{n} u_k^{\alpha_{ik}} > 0$, for $i = 1, ..., n; i \neq j$, and

$$\sum\nolimits_{j=1}^{n} u_j(x) \frac{\partial f_j}{\partial u_1} = \alpha_{11} u_1^{\alpha_{11}} \prod_{k=1,k\neq 1}^{n} u_k^{\alpha_{1k}} + ... + \alpha_{n1} u_n u_1^{\alpha_{n1}} \prod_{k=1,k\neq 1}^{n} u_k^{\alpha_{nk}}$$

$$> \alpha_{11} u_1^{\alpha_{11}} \prod_{k=1,k\neq 1}^{n} u_k^{\alpha_{1k}} > f_1(u_1, ..., u_n),$$

and

$$\sum\nolimits_{j=1}^{n} u_j(x) \frac{\partial f_j}{\partial u_2} = \alpha_{12} u_1 u_2^{\alpha_{12}-1} \prod_{k=1,k\neq 2}^{n} u_k^{\alpha_{1k}} + \alpha_{22} u_2^{\alpha_{22}} \prod_{k=1,k\neq 2}^{n} u_k^{\alpha_{2k}} + ... + \alpha_{n2} u_n u_2^{\alpha_{n2}} \prod_{k=1,k\neq 2}^{n} u_k^{\alpha_{nk}}$$

$$> \alpha_{22} u_2^{\alpha_{22}} \prod_{k=1,k\neq 2}^{n} u_k^{\alpha_{2k}} > f_2(u_1, ..., u_n).$$

Similarly for $i = 3, ..., n$, we obtain $\sum_{j=1}^{n} u_j(x) \frac{\partial f_j}{\partial u_i} > f_i(u_1, ..., u_n)$. Hence by Theorem 1., any positive stationary solution of Equation (26) is linearly unstable. Or since

$$\left(\frac{f_i(u_{1,...,u_n})}{u_i}\right)_{u_i} = (\alpha_{ii} - 1) u_i^{\alpha_{ii}-2} \prod_{k=1,k\neq i}^{n} u_k^{\alpha_{ik}} > 0,$$

for $i = 1, ..., n$, thus by Theorem 2, any positive stationary solution is linearly unstable.

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