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A Mixed Initial Boundary Value Problem for the Degasperis-Procesi Equation on a Bounded Interval

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Abstract: In this paper, we investigate the following mixed initial boundary value problem for the Degasperis-Procesi equation

$$\begin{cases} u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \ t > 0, \ x \in [0, \ 1] \\ u(t, \ 0) = u_x(t, \ 1) = 0, u(t, \ 1) = u_{xx}(t, \ 1), \ t > 0 \\ u(0, \ x) = u_0(x) \end{cases}.$$

We establish local well-posedness to it by using Kato's theorem for abstract quasilinear evolution equation of hyperbolic type. A blow-up result is also presented.

Keywords: Degasperis-Procesi equation; initial boundary value problem; blow up

1 Introduction

Recently, Degasperis and Procesi [1] derived a new nonlinear dispersive partial differential equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \tag{1.1}$$

which is called Degasperis-Procesi equation. Degasperis, Holm and Hone [2] showed that Eq.(1.1) is integrable by deriving a Lax pair and a bi-Hamiltonian structure for it. Dullin et al. [3] showed that the Eq.(1.1) can be regarded as a model for nonlinear shallow water dynamics. Yin proved local well-posedness to Eq.(1.1) with initial data $u_0 \in H^S(R)$, $S > 3$ on the line [4] and on the circle [5]. The global existence of strong solutions and weak solutions to Eq.(1.1) are investigated in [5]-[10]. For the blow-up of solution to Cauchy problem for Eq.(1.1) we refer to [8]-[11]. Vakhnenko and Parkes [12] obtained the traveling wave solutions to Eq.(1.1). The shock wave solutions to Eq.(1.1) were investigated in [13]-[16]. Lenells [17] classified all weak traveling wave solutions. For more exact solutions to Eq.(1.1), we refer to [18]-[21]. Tian and Ni [22] studied the local well-posedness to the Cauchy problem for Eq.(1.1) of general type. Escher and Yin [23] studied the initial boundary value problem for Eq.(1.1) on the half line R^+ and on the finite interval $[0, 1]$. On $[0, 1]$, they used the boundary conditions $u(t, \ 0) = u(t, \ 1) = 0$, and obtained the local existence of solution to Eq.(1.1). They also obtained that the solution $u(\cdot, u_0)$ to Eq.(1.1) with the boundary conditions $u(t, \ 0) = u(t, \ 1) = 0$ and the initial data $u_0 \in H^S(0,1) \cap H_0^1(0,1)$ ($\frac{3}{2} < S \leq 3$) blows up in finite time if $u_0 \neq 0$. But they didn't point out where the blow-up phenomenon occurs.

In this paper, we use the boundary conditions different from that used in [23] and investigate a mixed initial boundary value problem for the Eq.(1.1) as follows

$$\begin{cases} u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \ t > 0, \ x \in [0, \ 1] \\ u(t, \ 0) = u_x(t, \ 1) = 0, u(t, \ 1) = u_{xx}(t, \ 1), \ t > 0 \\ u(0, \ x) = u_0(x), \ x \in [0, \ 1] \end{cases} \tag{1.2}$$

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The remainder of the paper is organized as follows. In Section 2, by using Kato's theorem [24] for abstract quasilinear evolution equation of hyperbolic type, we prove that the closed-loop system (1.2) is locally well-posed. In Section 3, we show that the solution to closed-loop system (1.2) blows up in finite time exactly on the boundary: $x = 0$ if initial data $u_0(x)$ satisfies $u_{0x}(0) < 0$.

2 Local well-posedness

In the following, we denote by $L^P(0,1)$ the Lebesgue space for $1 \le P \le \infty$, and the norm in $L^P(0,1)$ will be written $\|\cdot\|_{L^P}$. We also denote by $H^S(0,1)$ the usual Sobolev space for $S > 0$, and the norm in $H^S(0,1)$ will be written $\|\cdot\|_{H^S}$. Set $H^2_{0,1}(0,1) = \{\phi \in H^2(0,1) : \phi(0) = \phi'(1)\}$, $H^1_1(0,1) = \{\phi \in H^1(0,1) : \phi(1) = 0\}$.

Theorem 2.1 *If $u_0(x) \in H^2_{0,1}(0,1) \cap H^3(0,1)$ and $u_0(x) - u_{0xx}(x) \in H^1_1(0,1)$, then there exists a maximal time $T = T(u_0) > 0$ such that problem (1.2) has a unique solution u satisfying*

$$u(t,x) = u(\cdot, u_0) \in C([0,T); H^2_{0,1}(0,1) \cap H^3(0,1)) \cap C^1([0,T); H^2(0,1)).$$

Moreover, the solution depends continuously on the initial data.

Proof. We'll apply the Kato's theorem to prove the local existence of the solution to problem (1.2). To this end, we rewrite (1.2) as the following equivalent form

$$\begin{cases} m_t + u m_x = -3u_x m \\ u(t,\ 0) = u_x(t,\ 1) = 0 \\ m(t,\ 1) = 0 \\ u(0,x) = u_0(x), m(0,x) = u_0(x) - u_{0xx}(x) \end{cases} \tag{2.1}$$

where $m = u - u_{xx}$.

Let $X = L^2(0,1)$, $Y = H^1_1(0,1)$. Obviously, Y is continuously and densely embedded in X. Let $L(Y,X)$ denotes the space of all bounded linear operators from Y to X. If $X = Y$, we denote this space by $L(X)$.

Set $S = I - \partial_x^2$, then $u(t,x) = S^{-1}m(t,x)$. So in Space $Y = H^1_1(0,1)$, problem (2.1) can be rewritten as the following Cauchy problem for an abstract quasilinear evolution equation

$$\begin{cases} m_t + (S^{-1}m)m_x = -3m(S^{-1}m)_x \\ m(0,x) = m_0(x) \end{cases} \tag{2.2}$$

where $m_0(x) = u_0(x) - u_{0xx}(x)$.

For each $m(t,x) \in Y$, the solution of equation $u - u_{xx} = m(t,x)$ in $H^2_{0,1}(0,1)$ satisfies

$$u(t,x) = S^{-1}m(t,x) = \frac{ch(1-x)}{ch1}\int_0^x m(t,\xi)sh\xi d\xi + \frac{shx}{ch1}\int_x^1 m(t,\xi)ch(\xi-1)d\xi$$

and

$$\left\|S^{-1}m\right\|^2_{L^2} + 2\left\|(S^{-1}m)_x\right\|^2_{L^2} + \left\|(S^{-1}m)_{xx}\right\|^2_{L^2} = \|m\|^2_{L^2} \tag{2.3}$$

$$\left\|S^{-1}m\right\|^2_{L^\infty} \le \|m\|^2_{L^2} \tag{2.4}$$

$$\left\|(S^{-1}m)_x\right\|^2_{L^\infty} \le \|m\|^2_{L^2} \tag{2.5}$$

$$S^{-1}m_x = (S^{-1}m)_x + \frac{1}{ch1}\int_0^1 ch(x+\xi-1)m(t,\xi)d\xi \tag{2.6}$$

We define the operator $A(m) := (S^{-1}m)\partial_x$. In view of (2.3) , we have

$$\|A(m)w\|_X = \left\|(S^{-1}m)w_x\right\|_X \le \|m\|_X \|w_x\|_X \le \|m\|_X \|w\|_Y, \forall w \in Y,$$

ie.,

$$\|A(m)\|_{L(Y,X)} \leq \|m\|_X\,,$$

so $A(m) \in L(Y,X)$. Moreover, it follows from (2.4) that

$$\begin{aligned}\|(A(y)-A(z))w\|_X &= \left\|(S^{-1}y-S^{-1}z)w_x\right\|_X\\ &\leq \left\|S^{-1}(y-z)\right\|_{L^\infty}\|w_x\|_X\\ &\leq \|y-z\|_X\|w\|_Y\,, \forall y,z,w\in Y.\end{aligned}$$

Note that for each $y \in Y$,

$$(A(m)w,w)_X = \int_0^1 (S^{-1}m)w_x w dx = -\frac{1}{2}\int_0^1 (S^{-1}m)_x w^2 dx. \tag{2.7}$$

If $\lambda(\gamma) = \max\limits_{x\in[0,1]} \frac{1}{2}\left|(S^{-1}m)_x\right|$, then by (2.5), we get $\lambda(\gamma) \leq \frac{1}{2}\|m\|_{L^2}$.
Using (2.7), we obtain

$$((A(m)+\beta)w,w)_X \geq (\beta-\lambda(\gamma))\|w\|_X^2\,, \quad \forall\beta \geq \lambda(\gamma).$$

This implies $A(m)$ is quasi-m-accretive uniformly on bounded sets in Y.

Define

$$B(m) := (m-S^{-1}m)(S^{-1})_x + 2(S^{-1}m)_x(I-S^{-1}),$$

where S^{-1} is the inverse operator of S in space$Y$. For each $w \in Y$,

$$\begin{aligned}SA(m)S^{-1}w = S(S^{-1}m)(S^{-1}w)_x &= m(S^{-1}w)_x - 2(S^{-1}m)_x(S^{-1}w-w)\\ &-(S^{-1}m)(S^{-1}w)_x + A(m)w,\end{aligned}$$

so

$$SA(m)S^{-1} - A(m) = (m-S^{-1}m)(S^{-1})_x + 2(S^{-1}m)_x(I-S^{-1}) \tag{2.8}$$

and

$$SA(m)S^{-1} = A(m) + B(m).$$

From (2.8) and the density of Y in X, we know $B(m)$ is a linear operator in X.
By (2.4), (2.5), we arrive at

$$\|B(m)w\|_X = \left\|(m-S^{-1}m)(S^{-1}w)_x + 2(S^{-1}m)(w-S^{-1}w)\right\|_X \leq 6\|m\|_X\|w\|_X\,. \tag{2.9}$$

This shows that $B(m) \in L(X)$ and $\|B(m)\|_{L(X)} \leq 6\|m\|_X$. So $B(m)$is bounded uniformly on bounded sets in Y. Moreover, by (2.9), we get

$$\begin{aligned}&\|(B(y)-B(z))w\|_X = \|B(y)w-B(z)w\|_X\\ &= \left\|(y-S^{-1}y)(S^{-1}w)_x + 2(S^{-1}y)(w-S^{-1}w) - (z-S^{-1}z)(S^{-1}w)_x - 2(S^{-1}z)(w-S^{-1}w)\right\|\\ &\leq \|y-z\|_X\|w\|_X\,, \quad y,z,w\in Y.\end{aligned}$$

Define $f(m) := -3m(S^{-1}m)_x$, then $f:\ Y \to Y$ extends to a map from X into X and $f(m)$is bounded on bounded sets in Y.
By (2.4), we have

$$\|f(m)\|_Y = \left\|-3(S^{-1}m)_x m\right\|_Y \leq 3\left\|(S^{-1}m)_x\right\|_{L^\infty}\|m\|_Y \leq 3\|m\|_X\|m\|_Y \leq 3\|m\|_Y^2$$

It follows from (2.4), (2.5) that

$$\begin{aligned}\|f(y)-f(z)\|_X &= \left\|-3y(S^{-1}y)_x - 3z(S^{-1}z)_x\right\|_X\\ &= 3\left\|(y-z)(S^{-1}y)_x + z(S^{-1}(y-z))_x\right\|_X\\ &\leq 3(\|y-z\|_X\left\|S^{-1}y\right\|_{L^\infty} + \|z\|_X\left\|(S^{-1}(y-z))_x\right\|_{L^\infty})\\ &\leq 3(\|y-z\|_X\|y\|_X + \|z\|_X\|y-z\|_X)\\ &= 3(\|y\|_X + \|z\|_X)\|y-z\|_X\,, \quad \forall y,z\in X.\end{aligned}$$

In view of (2.6), we get

$$f(m) = -3m(S^{-1}m)_x = -3m(S^{-1}m_x - \frac{1}{ch1}\int_0^1 ch(x+\xi-1)m(\xi)d\xi). \tag{2.10}$$

Define

$$C(m) := -\frac{1}{ch1}\int_0^1 ch(x+\xi-1)m(\xi)d\xi, \tag{2.11}$$

then $C(m)$ is a linear operator on X and

$$(C(m))_x = -\frac{1}{ch1}\int_0^1 sh(x+\xi-1)m(t,\xi)d\xi$$

$$\|C(m)\|_{L^\infty} \le \int_0^1 m(\xi)d\xi \le \|m\|_X \le \|m\|_Y\,, \tag{2.12}$$

$$\|(C(m))_x\|_{L^\infty} \le \int_0^1 m(\xi)d\xi \le \|m\|_X \le \|m\|_Y\,. \tag{2.13}$$

Then for $y, z \in X$, there are

$$f(y) - f(z) = -3[(y-z)(S^{-1}y_x + C(y)) + z(S^{-1}(y-z)_x + C(y-z))]$$

$$\begin{aligned}(f(y)-f(z))_x = -3[&(y-z)_x(S^{-1}y_x + C(y)) + (y-z)((S^{-1}y_x)_x + (C(y))_x \\ &+ z_x(S^{-1}(y-z)_x + C(y-z)) + z((S^{-1}(y-z)_x)_x + (C(y-z))_x)].\end{aligned} \tag{2.14}$$

From (2.10)-(2.13), we obtain

$$\|f(y)-f(z)\|_Y = \|f(y)-f(z)\|_X + \|(f(y)-f(z))_x\|_X \le 18(\|y\|_Y + \|z\|_Y)\,\|y-z\|_Y\,.$$

By Kato's theorem, there exists a constant $T > 0$, such that there is a unique solution

$$m = m(\cdot, m_0) \in C([0,T); H_1^1(0,1)) \cap C^1([0,T); L^2(0,1))$$

to the problem (2.1). Moreover, m depends continuously on m_0. Equivalently, the closed-loop system (1.2) has a unique solution

$$u(t,x) = u(\cdot, u_0) \in C([0,T); H_{0,1}^2(0,1) \cap H^3(0,1)) \cap C^1([0,T); H^2(0,1)),$$

and u depends continuously on u_0. ■

3 Blow up

Theorem 3.1 *Suppose that* $u_0(x) \in H_{0,1}^2(0,1) \cap H^3(0,1)$, $m_0(x) = u_0(x) - u_{0xx}(x) \in H_1^1(0,1)$ *and* $u(t,x) \in C([0,T); H_{0,1}^2(0,1) \cap H^3(0,1)) \cap C^1([0,T); H^2(0,1))$ *is the solution to problem (1.2) guaranteed by Theorem 2.1. If* $u_{0x}(0) = \frac{1}{ch1}\int_0^1 m_0(\xi)ch(\xi-1)d\xi < 0$, *then there exists a constant* $T_0 > 0$ *, such that* $\lim_{t\to T_0^-} u_x(0,t) = -\infty$.

Proof. Problem (1.2) can be rewritten as

$$\begin{cases} u_t + uu_x - (u_t + uu_x)_{xx} = -(\frac{3}{2}u^2)_x \\ u_t + uu_x|_{x=0} = 0, u_t + uu_x|_{x=1} = u^2(t,1) \end{cases} \tag{3.1}$$

Define R^{-1} as follows

$$R^{-1}(m(t,x)) = \frac{ch(1-x)}{ch1}\int_0^x m(t,\xi)sh\xi d\xi + \frac{shx}{ch1}[-u^2(t,1) + \int_x^1 m(t,\xi)ch(\xi-1)d\xi],$$

then from (3.1), we have

$$\begin{aligned} u_t + uu_x &= -R^{-1}((\frac{3}{2}u^2)_x) \\ &= -\frac{ch(1-x)}{ch1}\int_0^x (\frac{3}{2}u^2(t,\xi))_\xi sh\xi d\xi - \frac{shx}{ch1}(-u^2(t,1) + \int_x^1 (\frac{3}{2}u^2(t,\xi))_\xi ch(\xi-1)d\xi \\ &= \frac{ch(1-x)}{ch1}\int_0^x (\frac{3}{2}u^2(t,\xi))ch\xi d\xi + \frac{shx}{ch1}\int_x^1 (\frac{3}{2}u^2(t,\xi))sh(\xi-1)d\xi - \frac{shx}{2ch1}u^2(t,1) \end{aligned} \tag{3.2}$$

Differentiating (3.2) with respect to x, we obtain

$$\begin{aligned} (u_t + uu_x)_x &= u_{xt} + u_x^2 + uu_{xx} \\ &= -\frac{sh(1-x)}{ch1}\int_0^x \frac{3}{2}u^2(t,\xi)ch\xi d\xi \\ &+ \frac{chx}{ch1}\int_x^1 \frac{3}{2}u^2(t,\xi)sh(\xi-1)d\xi + \frac{3}{2}u^2(t,x) - \frac{chx}{2ch1}u^2(t,1). \end{aligned} \tag{3.3}$$

Let $x = 0$ in (3.3) and denote $h(t) = u_x(t,0)$, we have

$$h'(t) = -h^2(t) - \frac{1}{ch1}u^2(t,1) + \frac{1}{ch1}\int_0^1 \frac{3}{2}u^2(t,\xi)sh(\xi-1)d\xi.$$

Noting that $sh(\xi-1) < 0$ for $0 < \xi < 1$, we obtain $h'(t) < -h^2(t)$ if $h(0) = u_x(0,0) = u_{0x}(0) < 0$. Hence

$$-h'(t) > \frac{1}{-\frac{1}{h(0)} - t} \to +\infty \quad (t \to -\frac{1}{h(0)}),$$

so there exists a positive constant $T_0 \leq -\frac{1}{u_{0x}(0)}$, such that $\lim_{t\to T_0^-} h(t) = \lim_{t\to T_0^-} u_x(0,t) = -\infty$. ■

Remark 3.1 *The Cauchy problem for Eq.(1.1) discussed in [6] is global well-posed if $m_0(x) = u_0(x) - u_{0xx}(x)$ doesn't change sign. But Theorem 3.1 shows that the solution to the initial boundary value problem for Eq.(1.1) discussed in this paper (ie., problem (1.2)) blows up in finite time on the boundary $x = 0$ whenever $m_0(x) < 0$ or $u_{0x}(x) < 0$. This shows that the boundary conditions have important impact on Eq.(1.1) and our work of investigating the initial boundary problem for Eq.(1.1) is meaningful.*

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References

[1] A. Degasperis, M. Procesi: Asymptotic integrability, in: A. Degasperis, G. Gaeta (Eds.), *Symmetry and Perturbation Theor*y (SPT 98), Rome, December 1998, World Scientific, River Edge, NJ, 23-37 (1999)

[2] A. Degasperis, D. D. Holm, A. N. W. Hone: A new integral equation with peakon solutions. *Theo. Math.Phys.* 133, 1463–1474 (2002)

[3] H. R. Dullin, G. A. Gottwald, D. D. Holm: Camassa-Holm, Kortewed-de Vries-5 and other asymptotically equivalent equations for shallow water waves. *Fluid Dyn Res* 33, 73-95 (2003)

[4] Z. Yin: On the Cauchy problem for an integrable equation with peakon solutions. *Ill. J. Math.* 47(3), 649–666 (2003)

[5] Z. Yin: Global weak solutions to a new periodic integrable equation with peakon solutions. *J. Funct.Anal.* 212(1), 182–194 (2004)

[6] Z. Yin: Global solutions to a new integrable equation with peakons. *Ind. Univ. Math. J.* 53(4), 1189–1210 (2004)

[7] Z. Yin: Global existence for a new periodic integrable equation. *J. Math. Anal. Appl.* 283(1), 129-139 (2003)

[8] Y. Liu, Z. Yin: Global existence and blow-up phenomena for the Degasperis-Procesi equation. *Commun. Math. Phys.* 267, 801-820 (2006)

[9] Z. Yin: Well-posedness, blow up, and global existence for an integrable shallow water equation. *Disc. Conti. Dyn. Sys.* 11(2-3), 393-411 (2004)

[10] J. Escher, Y. Liu, Z. Yin: Global weak solutions and blow-up structure for the Degasperis-Procesi equation. *J. Func. Anal.* 241(2), 457-485 (2006)

[11] Y. Zhou: Blow-up phenomena for the integrable Degasperis-Procesi equation. *Phys. Lett.A* 328(2-3), 157-162 (2004)

[12] V. Vakhnenko, E. Parkes: Periodic and solitary-wave solutions of the Degasperis–Procesi equation. *Chaos Solitons & Fractals* 20(5), 1059-1073 (2004)

[13] H. Lundmark: Formation and dynamics of shock waves in the Degasperis-Procesi equation. *J. Nonlinear Sci.* 17, 169-198 (2007)

[14] J. Escher, Y. Liu, Z. Yin: Shock waves and blow-up phenomena for the periodic degasperis equation. *Ind. Univ. Math. J.* 56, 187-117 (2007)

[15] G. M. Coclite, K. H. Karlsen, N. H. Risebro: Numerical schemes for computing discontinuous solutions of the Degasperis-Procesi equation. *IMA J. Numer. Anal*. doi: 10. 1093/imanum/drum003 (2007)

[16] G. M. Coclite, K. H. Karlsen, N. H. Risebro: On the uniqueness of discontinuous solutions to the Degasperis-Procesi equation. *J. Diff. Equ.* 234(1), 142-160 (2007)

[17] J. Lenells: Traveling wave solutions of the Degasperis-Procesi equation. *J. Math. Anal. Appl.* 306(1), 72-82 (2005)

[18] Y. Matsuno: Multisoliton solutions of the Degasperis-Procesi equation and their peakon limit. *Inverse Problems* 21(5), 1553-1570 (2005)

[19] H. Lundmark, J. Szmigielski: Multi-peakon solutions of the Degasperis-Procesi equation. *Inverse Problems* 19(6), 1241-1245 (2003)

[20] Y. Matsuno: Cusp and loop soliton solutions of short-wave models, for the Camassa-Holm and Degasperis-Procesi equations. *Physics Letters A* 359(5), 451-457 (2006)

[21] L. Yu, L. Tian, X. Wang: The bifurcation and peakon for Degasperis-Procesi equation. *Chaos, Solitons & Fractals* 30(4), 956-966 (2006)

[22] L.Tian, X. Li: On the well-posedness problem for the generalized Degasperis-Procesi equation. *International Journal of Nonlinear Science* 2(2), 67-76 (2006)

[23] J. Escher, Z. Yin: On the initial boundary value problems for the Degasperis-Procesi equation. *Physics Letters A* 368(1-2), 69-76 (2007)

[24] T. Kato: Quasi-linear equations of evolution, with applications to partial differential quations. In: Spectral Theory and Differential Equations, *Lecture Notes in Math.* 448. Berlin: Springer Verlag, 25-70 (1975)