ACADIMIC World Academic Union

ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.5(2008) No.3, pp. 203-209

An Approximation of the Analytic Solution of the Cauchy Problem Arising in One Dimensional Nonlinear Thermoelasticity

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(Received 6 May 2007, accepted 19 December 2007)

Abstract: In this work, we will discuss the solution of a Cauchy problem arising in one dimensional nonlinear thermoelasticity. The main objective is to propose an alternative method of solution, one not based on finite difference or finite element or spectral methods. The aim of the present paper is to investigate the application of Adomian decomposition method for solving a one dimensional nonlinear thermoelasticity. This method needs less work in comparison with the traditional methods and decreases considerable volume of calculation. In this work, we are concerned with the application of the decomposition method to the thermoelasticity problem. The comparison of the numerical solutions obtained by this method with the exact solution shows the efficiency of this method.

Keywords: Adomian decomposition method; analytic solution; Cauchy problem; nonlinear thermoelasticity

1 Introduction

Most scientific problems and phenomena in different fields of science and engineering occur nonlinearly, especially in fluid mechanics, solid state physics, plasma physics, chemical physics, plasma waves and thermoelasticity. Except in a limited number of these problems, we encounter difficulties in finding their exact analytical solutions. Thermoelasticity problems have gained a considerable attention for their importance and applications. Much effort was paid on existence of the solution of thermoelasticity problems and various numerical techniques were used[1-8].

Recently, numerical methods, which do not require discretization of space-time variables or linearization of the nonlinear equations, are introduced for finding analytical solutions of nonlinear problems, among which Adomian decomposition method (ADM) [9-17] is one of the most effective, convenient and accurate one for both weakly and strongly nonlinear problems. Some other methods are reviewed in Refs. [18-24]. ADM is to split the given equation into linear and nonlinear parts, invert the highest-order derivative operator contained in the linear operator in both sides, calculate Adomian's polynomials, and finally find the successive terms of the series solution by recurrent relation using Adomian's polynomials [25,26].

The aim of this work is to employ ADM to solve a real-life problem that exhibits coupling between the mechanical and thermal fields. Let consider the following nonlinear system arising in thermoelasticity [2, 5, 6]:

$$u_{tt} - a(u_x, \theta)u_{xx} + b(u_x, \theta)\theta_x = f(x, t) \tag{1}$$

$$c(u_x, \theta)\theta_t + b(u_x, \theta)u_{xt} - d(\theta)\theta_{xx} = g(x, t) \tag{2}$$

subject to the initial conditions of:

$$u(x, 0) = u^0(x), u_t(x, 0) = u^1(x), \theta(x, 0) = \theta^0(x) \tag{3}$$

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IJNS.2008.06.15/144

where $u(x,t)$ is the body displacement form equilibrium and $\theta(x,t)$ is the difference of the body's temperature from a reference $T_0 = 0$, subscripts denote partial derivatives and a, b, c and d are given smooth functions. For more details about the physical meaning of the model, see [6,27].

In order to illustrate the effectiveness of the method, an artificial model is used. Let us define a, b, c, d, u^0, u^1 and θ^0 by [5]:

$$a(u_x,\theta) = 2 - u_x\theta, b(u_x,\theta) = 2 + u_x\theta, c(u_x,\theta) = 1, c(u_x,\theta) = 0 \quad (4)$$

$$u_x^0 = \frac{1}{1+x^2}, u_x^1 = 0, \theta_x^0 = \frac{1}{1+x^2} \quad (5)$$

And replace the right-hand side of above equations by:

$$f(x,t) = \frac{2}{1+x^2} - \frac{2(1+t^2)(3x^2-1)}{(1+x^2)^3}a(w,v) - \frac{2x(1+t)}{(1+x^2)^2}b(w,v) \quad (6)$$

$$g(x,t) = \frac{1}{1+x^2}c(w,v) - \frac{4xt}{(1+x^2)^2}b(w,v) - \frac{2(1+t)(3x^2-1)}{(1+x^2)^3}d(v) \quad (7)$$

$$w \equiv w(x,t) = \frac{-2x(1+t^2)}{(1+x^2)^2}, v \equiv v(x,t) = \frac{1+t}{1+x^2} \quad (8)$$

where a, b, c and d are defined by Eq.(4) and The exact solution of first two equations is given by[5]:

$$u(x,t) = \frac{1+t^2}{(1+x^2)^2}, \theta(x,t) = \frac{1+t}{1+x^2} \quad (9)$$

In the next section, a brief outline of ADM is explained. Then, ADM is implemented to the nonlinear thermoelastic system. Finally, the results obtained by ADM and the exact solution are compared graphically.

2 Fundamentals of Adomian decomposition method

Let us discuss a brief outline of the Adomian Decomposition method. For this, we consider a general nonlinear equation in the form:

$$Lu + Ru + Nu = g \quad (10)$$

where L is the highest order derivative which is assumed to be easily invertible, R the linear differential operator of less order than L ,Nu presents the nonlinear terms and g is the source term. Applying the inverse operator L^{-1} to the both sides of Eq. (10), and using the given conditions we obtain:

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Lu) \quad (11)$$

where the function f(x) represents the terms arising from integration the source term g(x) , using given conditions. For nonlinear differential equations, the nonlinear operator Nu=f(x) is represented by an infinite series of the so-called Adomian polynomials

$$F(u) = \sum_{m=0}^{\infty} A_m \quad (12)$$

The polynomials A_m are generated for all kind of nonlinearity so that A_0 depends only on u_0 ,$A1$ depends on u_0 and u_1 , and so on. The Adomian polynomials introduced above show that the sum of subscripts of the components of u for each term of A_m is equal to [25].

The Adomian method defines the solution $u(x)$ by the series

$$u = \sum_{m=0}^{\infty} u_m \quad (13)$$

In the case of F_u, the infinite series is a Taylor expansion about u_0 , as follows:

$$F(u) = F(u_0) + F'(u_0)(u - u_0) + F''(u_0)\frac{(u - u_0)}{2!} + F'''(u_0)\frac{(u - u_0)^2}{3!} + ... \quad (14)$$

By rewriting Eq. (13) as $u - u_0 = u_1 + u_2 + u_3 + ...$, substituting it into Eq. (14) and then equating two expressions for $F(u)$ found in Eq. (14) and Eq. (12), defines formulas for the Adomian polynomials in the form of

$$F(u) = A_1 + A_2 + ... = F(u_0) + F'(u_0)(u - u_0) + F''(u_0)\frac{(u - u_0)}{2!} + ... \quad (15)$$

By equating terms in Eq. (9), the first few Adomian's polynomials A_0 ,A_1 ,A_2 ,A_3 and A_4 are given

$$A_0 = F(u_0) \quad (16)$$

$$A_1 = u_1 F'(u_0) \quad (17)$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2} u_1^2 F''(u_0) \quad (18)$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3} u_1^3 F'''(u_0) \quad (19)$$

$$A_4 = u_4 F'(u_0) + (\frac{1}{2!} u_2^2 + u_1 u_3) F''(u_0) + \frac{1}{2!} u_1^2 u_2 F'''(u_0) + \frac{1}{4!} F^{(iv)}(u_0) \quad (20)$$

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Now that the A_m are known , Eq. (12) can be substituted in Eq.(11) to specify the terms in the expansion for the solution of Eq. (13).

3 Implementation of ADM to thermoelasticity problem

In order to apply ADM to nonlinear thermoelasticity problem, we rewrite Eqs. (1) and (2) in the following operator form:

$$L_{tt} u = 2L_{xx} u - 2L_x \theta - u_x u_{xx} \theta - u_x \theta \theta_x + f(x, t) \quad (21)$$

$$L_t \theta = -2L_{xt} u - u_x u_{xt} \theta + \theta\theta(x) + g(x, t) \quad (22)$$

where the notations

$$L_{tt} = \frac{\partial^2}{\partial t^2}, L_{xx} = \frac{\partial^2}{\partial x^2}, L_t = \frac{\partial}{\partial t}, L_x = \frac{\partial}{\partial x}, L_{xt} = \frac{\partial^2}{\partial x \partial t} \quad (23)$$

are the linear operators. By using the inverse operators, we can write Eqs. (21) and (22) in the following form:

$$u(x, t) = u(x, 0) + t u_t(x, 0) + L_{tt}^{-1}[2L_{xx} u - 2L_x \theta - u_x u_{xx} \theta - u_x \theta \theta_x + f(x, t)] \quad (24)$$

$$\theta(x, t) = \theta(x, 0) + L_t^{-1}[-2L_{xt} u - u_x u_{xt} \theta + \theta\theta(x) + g(x, t)] \quad (25)$$

where the inverse operators are defined by

$$L_{tt}^{-1}(\square) = \int_0^t \int_0^t (\square) dt dt, L_t^{-1}(\square) = \int_0^t (\square) dt \quad (26)$$

Using Eq. (5), we obtain

$$u(x, t) = u(x, 0) + L_{tt}^{-1}[2L_{xx} u - 2L_x \theta - N_1(u, \theta) - N_2(u, \theta) + f(x, t)] \quad (27)$$

$$\theta(x, t) = \theta(x, 0) + L_t^{-1}[-2L_{xt} u - N_3(u, \theta) + N_4(u, \theta) + g(x, t)] \quad (28)$$

The solutions $u(x,t)$ and $\theta(x,t)$ can be decomposed by an infinite series, as follows:

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t),\theta(x,t)=\sum_{i=0}^{\infty}\theta_i(x,t) \tag{29}$$

The nonlinear operators $N_1(u,\theta)$, $N_2(u,\theta)$, $N_3(u,\theta)$, $N_4(u,\theta)$ are defined by the following infinite series [28]:

$$N_i(u,\theta)=\sum_{n=0}^{\infty}A_{in}, i=1,2,3,4 \tag{30}$$

where A_{in} is called Adomian polynomials [25, 26, 28] and defined by:

$$A_{in}=\frac{1}{n!}[\frac{d^n}{d\lambda^n}N_i(\sum_{k=0}^{n}\lambda^k u_k,\sum_{k=0}^{n}\lambda^k\theta_k)]_{\lambda=0} \tag{31}$$

Hence we obtain the components series solution by the following recursive relations:

$$u_0(x,t)=u(x,0),\theta_0(x,t)=\theta(x,0) \tag{32}$$

$$u_{n+1}=L_{tt}^{-1}[2L_{xx}u_n-2L_x\theta_n-A_{1,n}-A_{2,n}+f(x,t)] \tag{33}$$

$$\theta_{n+1}=Lt^{-1}[-2L_{xt}u_n-A3,n+A4,n+g(x,t)] \tag{34}$$

where $n\geq 0$. Adomian's polynomials formula, Eq. (31), is easy to set computer code to get as many polynomials as we need in the calculation. We can give the first few Adomian's polynomials of the A_{in} as:

$$A_{1,0}=u_{0x}u_{0xx}\theta_0 \tag{35}$$

$$A_{1,1}=u_{0x}u_{0xx}\theta_1+u_{0x}u_{1xx}\theta_0+u_{1x}u_{0xx}\theta_0 \tag{36}$$

$$A_{1,2}=u_{1x}u_{1xx}\theta_0+u_{0x}u_{0xx}\theta_2+u_{0x}u_{1xx}\theta_1+u_{1x}u_{0xx}\theta_1+u_{2x}u_{0xx}\theta_0 \tag{37}$$

$$A_{2,0}=u_{0x}\theta_0\theta_{0x} \tag{38}$$

$$A_{2,1}=u_{0x}\theta_0\theta_{1x}+u_{0x}\theta_1\theta_{0x}+u_{1x}\theta_0\theta_{0x} \tag{39}$$

$$A_{2,2}=u_{0x}\theta_0\theta_{2x}+u_{0x}\theta_1\theta_{1x}+u_{0x}\theta_2\theta_{0x}+u_{1x}\theta_0\theta_{1x}+u_{1x}\theta_1\theta_{0x}+u_{2x}\theta_0\theta_{0x} \tag{40}$$

$$A_{3,0}=u_{0x}u_{0xt}\theta_0 \tag{41}$$

$$A_{3,1}=u_{0x}u_{0xt}\theta_1+u_{0x}u_{1xt}\theta_0+u_{1x}u_{0xt}\theta_0 \tag{42}$$

$$A_{3,2}=u_{0x}u_{0xt}\theta_2+u_{0x}u_{1xt}\theta_1+u_{0x}u_{2xt}\theta_0+u_{1x}u_{0xt}\theta_1+u_{1x}u_{1xt}\theta_1+u_{2x}u_{0xt}\theta_0 \tag{43}$$

$$A_{4,0}=\theta_0\theta_{0xx} \tag{44}$$

$$A_{4,1}=\theta_0\theta_{1xx}+\theta_1\theta_{0xx} \tag{45}$$

$$A_{4,2}=\theta_0\theta_{2xx}+\theta_2\theta_{0xx}+\theta_1\theta_{1xx} \tag{46}$$

and so on, the rest of the polynomials can be constructed in a similar manner. Using the recursive relations, Eqs. (33) and (34), and Adomian's polynomials formula, Eq. (31), with the initial conditions, Eq. (3), gives:

$$u_0(x,t)=\frac{1}{1+x^2} \tag{47}$$

$$u_1(x,t)=\frac{1}{105(1+x^2)^6}(10t^7(x-3x^3)+14t^6(x+x^2-3x3+x^4)+42t^5(x+x^2-3x^3+x^4) \tag{48}$$

$$+35t^4(1+x+2x^2-6x^3-2x^4-8x^6-3x^8)+70t^3(2x^2-7x^3+2x^4-6x^5-4x^7-x^9)$$

$$+105t^2(1+5x^2+10x^4+10x^6+5x^8+x^{10}))$$

$$\theta_0(x,t)=\frac{1}{1+x^2} \tag{49}$$

$$\theta_1(x,t)=\frac{1}{15(1+x^2)^5}(24t^5x^2+30t^4x^2+10t^3(1+2x^2-3x^4)+30t^2(1-2x-6x^3-3x^4-6x^5-2x^7 \quad (50)$$
$$+15t(1+4x^2+6x^4+4x^6+x^8)))$$

Proceeding in the same way, we can obtain $u_2(x,t)$, $\theta_2(x,t)$, and higher order approximations. Here, the numerical results are evaluated using terms approximation of the recursive relations.

Fig. 1 shows the comparison of the $u(x,t)$ obtained by ADM with that of the exact solution at various values of time. It is apparent that the results are in good agreement and the solution by ADM converges to that of the exact solution. The absolute errors of ADM at various values of time are also depicted in Fig. 2. A comparison of the $\theta(x,t)$ obtained by ADM with that of the exact solution at various values of time is shown in Fig. 3, which admits the efficiency and reliability of the method. The convergence of the solution by ADM is obviously seen in Fig. 4.

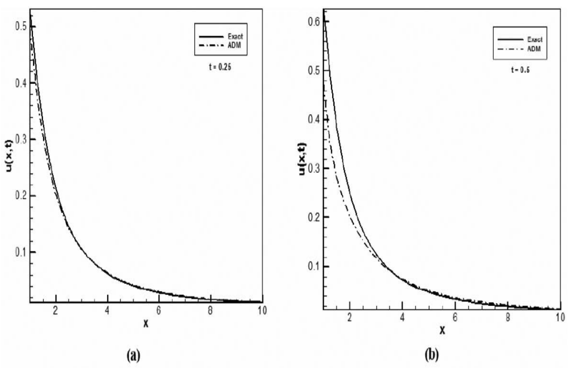

Figure 1: The comparison of the $u(x,t)$ obtained by ADM with the exact solution at (a):$t = 0.25$ (b):$t = 0.5$

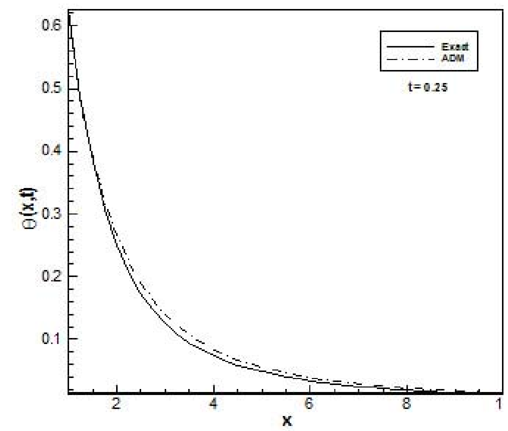

Figure 2: Absolute errors of ADM at various values of time

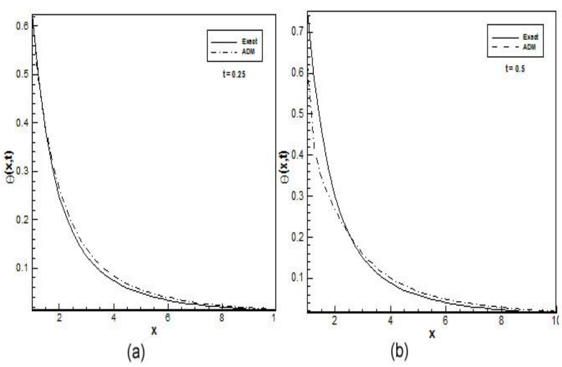

Figure 3: The comparison of the $\theta(x,t)$ obtained by ADM with the exact solution at (a):$t = 0.25$ (b):$t = 0.5$

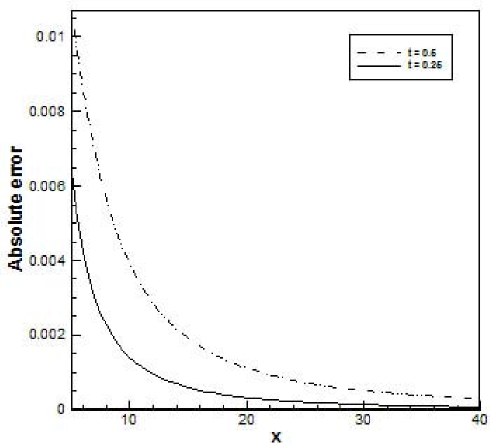

Figure 4: Absolute errors of ADM at various values of time

4 Conclusion

In this paper, we have successfully developed ADM to obtain an approximation of the analytic solution of the Cauchy problem arising in one dimensional nonlinear thermoelasticity. It is apparently seen that the method does not require complex calculations in comparison with the traditional methods, decreases considerable volume of calculations and avoids linearization and physically unrealistic assumptions. The decomposition procedure of Adomian will be obtained easily without linearization of the problem. In this approach the solution is found in the form of a convergent series with easily computed components. The results obtained by decomposition method are compared with those of the exact solution, which shows very good agreement, even using only few terms of the recursive relations. In general, this method provides highly accurate numerical solutions and can be applied to wide class of nonlinear problems in science and engineering.

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