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Global Conservative Solutions of the Generalized Camassa-Holm Equation

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Abstract: In this paper, we prove the existence of global conservative solutions of the Cauchy problem for the generalized Camassa-Holm equation. We transform it into an ODE system in a Banach space. By using the ODE theories and some related knowledge we obtain the existence of the short -time solutions. Particularly we obtain the global conservative solutions with respect to the initial date.

Keywords:generalized Camassa-Holm equation; global conservative solutions; Lipschitz

1 Introduction

In [1], Degasperis and Proesi studied the following family of third order dispersive PDE conservation laws,

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x \tag{1.1}$$

where α, c_0, c_1, c_2 and c_3 are real constants. They found that there are at least four equations that satisfy the completely integrability condition within this family: KdV equation, Camassa- Holm equation, Dullin-Gottwald-Holm equation and Degasperis- Procesi equation.

With $\alpha = c_2 = c_3 = 0$ in (1.1), it becomes the well-known Korteweg-de Veris equation.

The KdV equation is completely integrable and its solitary waves are solitions [2, 3]. The Cauchy problem of the KdV equation has been studied extensively, and a satisfactory local or global existence theory is proved in [4].

For $c_1 = -\frac{3}{2}c_3/\alpha^2$, $c_2 = c_3/2$, (1.1) becomes the Camassa-Holm equation.

$$u_t - u_{xxt} + u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.2}$$

It has a bi-Hamiltionian structure and is completely integrable (see [5]). In [6] Dangping Ding and Lixin Tian researched solution of dissipative Camassa-Holm equation on total space. Tian, Song, Yin [7, 8] considered the generalized Camassa-Holm equation and derived some new exact peakon and compacton.

Dullin, Gottwald,Holm [9] discussed the following $1 + 1$ quadratically nonlinear equation in this class for a unidirectional water wave with fluid velocity $u\,(x, t)$.

$$m_t + c_0 u_x + u m_x + 2m u_x = -\gamma u_{xxx}, \quad x \in R, \quad t \in R, \tag{1.3}$$

In [10, 11] Lixin Tian, Guilong Gui and Yue Liu studied the well-posedness of the Cauchy problem and the scattering problem for DGH equation.

With $c_1 = -2c_3/\alpha^2$, $c_2 = c_3$ in Eq.(1.1) , we find the Degasperis-Procesi equation of the form

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, t > 0\ , \quad x \in R \tag{1.4}$$

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Degasperis, Holm and Hone [12] proved the integrability of (1.4) by constructing a Lax pair. They also showed that Eq.(1.4) has bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admits exact peakon solutions which are analogous to the Camassa-Holm equation. After the Degasperis-Procesi Eq.(1.4) was derived, many papers were devoted to its study. For example, Yin [13] proved local well-posedness to Eq.(1.4) with initial data $u_0 \in H^s(R)$, $s > \frac{3}{2}$ and derived the precise blow-up scenario and a blow-up result. The global existence of strong solutions and global weak solutions to Eq. (1.4) was also investigated in [14, 15, 16, 17].

For the nonlinear partial differential equation

$$u_t - u_{xxt} - \gamma u_{xxx} + f(u)_x - f(u)_{xxx} + \left(g(u) + \frac{1}{2}f''(u)(u_x)^2\right)_x = 0$$

When $f(u) = \frac{1}{2}\gamma u^2$, $g(u)$ contains u^n $(n \geqslant 2)$ term, and we add a dispersive term γu_x , then we get the generalized Camassa-Holm equation

$$u_t - u_{xxt} + \frac{1}{2}g(u)_x - \gamma(2u_x u_{xx} + u u_{xxx} - u_x) = 0 \tag{1.5}$$

This is the equation which we will consider in this paper. When $g(u) = \frac{3-\gamma u^2}{2}$, Eq.(1.5) becomes Eq.(1.2). Here we take a different approach, based on recent techniques see [18, 19, 20, 21]. The equation can be reformulated as a system of ordinary differential equations taking values in a Banach space. In the space, we consider the conservative solutions that preserve the energy. We prove Eq.(1.5) possesses a global conservative solution. Furthermore, we show that the problem is well-posed.

This paper is organized as follows: In Section 2, first we transform the PDE into an ODE system. Short-time existence is derived by a contraction argument, see Theorem 2.3. Global existence with respect to both initial data and functions f and g , is obtained for a class of initial data that includes initial data $u|_{t=0} = \bar{u}$ in $H^1(\mathbb{R})$, see Theorem 2.7.

2 Existence of solutions

2.1 Transport equation for the energy density and reformulation in terms of Lagrangian variables

Eq.(1.5) is rewritten as the following term (see [22,23])

$$u_t + \gamma u u_x + P_x = 0, P - P_{xx} = \frac{1}{2}(g(u) - \gamma u^2 + \gamma u_x^2 + 2\gamma u) \tag{2.1}$$

It is advantageous to rewrite the equation as

$$u_t + f(u)_x + P_x = 0 \tag{2.2a}$$

$$P - P_{xx} = g(u) + \frac{1}{2}f''(u)u_x^2 + f''(u)u \tag{2.2b}$$

where we assume

$$\begin{cases} f \in W_{loc}^{2,\infty}, f''(u) \neq 0, u \in \mathbb{R} \\ g \in W_{loc}^{1,\infty}, g(0) \neq 0 \end{cases} \tag{2.3}$$

In (2.2 b), P can be written in explicit form:

$$P(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}\left(g \circ u + \frac{1}{2}f'' \circ u u_x^2 + f'' \circ u u\right)(t,z)dz \tag{2.4}$$

After differentiating (2.2a) with respect to x and using (2.2b), that

$$u_{xt} + f''(u)u_x^2 + f'(u)u_{xx} + P - g(u) - f''(u)u = 0 \tag{2.5}$$

Multiply (2.2a) by u , (2.5) by u_x , add the two to find the following equation

$$(u^2+u_x^2)_t+(f'(u)(u^2+u_x^2))_x=-2(Pu)_x+(2g(u)+f''(u)u^2+2f''(u)u)u_x \tag{2.6}$$

Define

$$G(v)=\int_0^v(2g(z)+f''(z)(z^2+2z))dz \tag{2.7}$$

Then (2.6) can be rewritten as

$$(u^2+u_x^2)_t+(f'(u)(u^2+u_x^2))_x=(G(u)-2Pu)_x \tag{2.8}$$

which is transport equation for the energy density $u^2+u_x^2$.

Define

$$y_t(t,\xi)=f'(u(t,y(t,\xi))) \tag{2.9}$$

Let the characteristics $y(t,\xi)$ are the solutions of(2.9), suppose $y(0,\xi)$ is given. Given ξ_1,ξ_2 in $\mathbb{R}$, let $H(t)=\int\limits_{y(t,\xi_1)}^{y(t,\xi_2)}(u^2+u_x^2)dx$ be the energy contained between the two characteristic curves $y(t,\xi_1)$, $y(t,\xi_2)$. Then, we have

$$\frac{dH}{dt}=\left[y_t(t,\xi)(u^2+u_x^2)\circ y(t,\xi)\right]_{\xi_1}^{\xi_2}+\int_{y(t,\xi_1)}^{y(t,\xi_2)}(u^2+u_x^2)_tdx. \tag{2.10}$$

We use (2.8) and (2.10) then integrate by parts, then we get

$$\frac{dH}{dt}=[(G(u)-2Pu)\circ y]_{\xi_1}^{\xi_2} \tag{2.11}$$

We now derive a system equivalent to (2.2). The calculations here are formal and will be justified later. Let y still denote the characteristics. We introduce two other variables, the Lagrangian velocity U and cumulative energy distribution H defined by

$$U(t,\xi)=u(t,y(t,\xi)), \tag{2.12}$$

$$H(t,\xi)=\int_{-\infty}^{y(t,\xi)}(u^2+u_x^2)dx \tag{2.13}$$

From the definition of the characteristics, it follows from (2.2a) that

$$U_t(t,\xi)=u_t(t,y)+y_t(t,\xi)u_x(t,y)=(u_t+f'(u)u_x)\circ y(t,\xi)=-P_x\circ y(t,\xi) \tag{2.14}$$

This last term can be expressed uniquely in term of y , U , and H . Namely, we have

$$P_x\circ y(t,x)=-\frac{1}{2}\int_{\mathbb{R}}sgn(y(t,\xi)-z)e^{-|y(t,\xi)-z|}\left(g\circ u+\frac{1}{2}f''(u)(u_x^2+2u)\right)(t,z)dz$$

After the change of variable $z=y(t,\eta)$,

$$\begin{gathered}P_x\circ y(t,x)=-\tfrac{1}{2}\int_{\mathbb{R}}sgn(y(t,\xi)-y(t,\eta))e^{-|y(t,\xi)-y(t,\eta)|}\\ \times\left(g\circ u+\tfrac{1}{2}f''(u)(u_x^2+2u)\right)(t,y(t,\eta))y_\xi(t,\eta)d\eta\end{gathered}$$

Finally, since $H_\xi=(u^2+u_x^2)\circ yy_\xi$,

$$\begin{gathered}P_x\circ y(t,\xi)=-\tfrac{1}{2}\int_{\mathbb{R}}sgn(y(\xi)-y(\eta))e^{-|y(\xi)-y(\eta)|}\\ \times\left((g(U)-\tfrac{1}{2}f''(U)U^2+f''(U)U)y_\xi+\tfrac{1}{2}f''(U)H_\xi\right)(\eta)d\eta\end{gathered} \tag{2.15}$$

Then $P_x \circ y$ is equivalent to Q where

$$Q(t,\xi) = -\frac{1}{2}\int_{\mathbb{R}} sgn(\xi-\eta)\exp(-sgn(\xi-\eta)(y(\xi)-y(\eta))) \\ \times\left((g(U)-\frac{1}{2}f''(U)U^2+f''(U)U)y_\xi+\frac{1}{2}f''(U)H_\xi\right)(\eta)d\eta \tag{2.16}$$

Slightly abusing the notation, we write

$$P(t,\xi) = \frac{1}{2}\int_{\mathbb{R}} \exp(-sgn(\xi-\eta)(y(\xi)-y(\eta))) \\ \times\left((g(U)-\frac{1}{2}f''(U)U^2+f''(U)U)y_\xi+\frac{1}{2}f''(U)H_\xi\right)(\eta)d\eta \tag{2.17}$$

$P_x \circ y$ and $P \circ y$ can be replaced by equivalent expressions given by (2.16) and (2.17) which only depend on our new variables U , H , and y . We introduce yet another variable $\zeta(t,\xi)$, simply defined as $\zeta(t,\xi) = y(t,\xi) - \xi$

It will turn out that $\zeta \in L^\infty(\mathbb{R})$. We have now derived a new system of equations, which is equivalent to (1. 5). Equations (2.9), (2.11) and (2.14) give us

$$\begin{cases} y_t = f'(u), \\ U_t = -Q, \\ H_t = G(U) - 2PU \end{cases} \tag{2.18}$$

Detailed analysis will reveal that the system (2.18) of ordinary differential equations for

$(\zeta,U,H) : [0\ ,T\ ] \to E$ is well-posed, where E is a Banach space to be defined in the next section. We have

$$Q_\xi = -\frac{1}{2}f''(U)H_\xi + \left(P + \frac{1}{2}f''(U)U^2 - f''(U)U - g(U)\right)y_\xi$$

and

$$P_\xi = Qy_\xi.$$

Then differentiating (2.18) yields

$$\begin{cases} \zeta_{\xi t} = f''(U)U_\xi, (y_{\xi t} = f''(U)U_\xi) \\ U_{\xi t} = \frac{1}{2}f''(U)H_\xi - \left(P + \frac{1}{2}f''(U)U^2 - f''(U)U - g(U)\right)y_\xi \\ H_{\xi t} = \left(2g(U) + f''(U)U^2 + 2f''(U)U - 2P\right)U_\xi - 2QUy_\xi \end{cases} \tag{2.19}$$

2.2 Existence and uniqueness of solutions in Lagrangian variables

Let V be the Banach space defined by $V = \left\{f \in C_b(\mathbb{R}) | f_\xi \in L^2(\mathbb{R})\right\}$, where $C_b(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and the norm of V is given by $\|f\|_V = \|f\|_{L^\infty(\mathbb{R})} + \|f_\xi\|_{L^2(\mathbb{R})}$.

Of course $H^1(\mathbb{R}) \subset V$, but the converse is not true as V contains functions that do not vanish at infinity. We will employ the Banach space E defined by $E = V \times H^1(R) \times V$ to carry out the contraction map argument.

For any $X = (\zeta,U,H) \in E$, the norm on E is given by

$$\|X\|_E = \|\zeta\|_V + \|U\|_{H^1(R)} + \|H\|_V .$$

In this section, we focus our attention on the system of Eqs. (2.18) and prove, by a contraction argument, that it admits a unique solution.

Lemma 2.1 *([26]) For any $X = (\zeta,U,H)$ in E , we define the maps Q and P as $Q(X) = Q$ and $P(X) = P$ where Q and P are given by (2.16) and (2.17). Then P and Q are locally Lipschitz maps from E to $H^1(\mathbb{R})$. Moreover,*

$$Q_\xi = -\frac{1}{2}f''(U)H_\xi + \left(P + \frac{1}{2}f''(U)U^2 - f''(U)U - g(U)\right)y_\xi, \\ P_\xi = Q(1+\zeta_\xi)$$

Lemma 2.2 *([26]) Let* $B_M = \{X \in E | \|X\|_E \leqslant M\}$ *(i) If* g_1 *is Lipschitz from* B_M *to* $L^\infty(\mathbb{R})$ *and* g_2 *is Lipschitz from* B_M *to* $L^2(\mathbb{R})$ *, then the product* $g_1 g_2$ *is Lipschitz from* B_M *to* $L^2(\mathbb{R})$ *. (ii) If* g_1, g_2, g_3 *are three Lipschitz maps from* B_M *to* $L^\infty(\mathbb{R})$ *, then the product* $g_1 g_2 g_3$ *is Lipschitz from* B_M *to* $L^\infty(\mathbb{R})$ *.*

Next we will use a contraction argument to prove the short-time existence of solutions to (2.18).

Theorem 2.1 $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{H})$ *in E, there exists a time T depending only on* $\|X\|_E$ *such that the system (2.18) admits a unique solution in* $C^1([0,T], E)$ *with initial data* $\bar{X}$*.*

Proof. For any $X = (\zeta, U, H)$, $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{H})$ *in* B_M , from (2.18) we know

$$X(t) = \bar{X} + \int_0^t F(X(\tau))d\tau \tag{2.20}$$

For $\|U\|_{L^\infty(\mathbb{R})} \leqslant lM, \|\bar{U}\|_{L^\infty(\mathbb{R})} \leqslant lM$,

$$\left\|f'(U) - f'(\bar{U})\right\|_{L^\infty(\mathbb{R})} \leqslant \|f'\|_{W^{1,\infty}} \|U - \bar{U}\|_{L^\infty(\mathbb{R})} \leqslant lL_M \|X - \bar{X}\|_E .$$

(l is the Lipschitz constant.)

Since $f'(U)_\xi = f''(U)U_\xi$, we know $X \to f'(U), X \to U_\xi$, are both Lipschitz on B_M , so $X \to f''(U)$ is Lipschitz from B_M into V , see [24]. And $X \to U_\xi$ is Lipschitz from B_M into $L^2(\mathbb{R})$, by Lemma 2.2 , $X \to G(U)$ is Lipschitz from B_M into $L^2(\mathbb{R})$.

$$F(X) = \left(f'(U), -Q(X), G(U) - 2P(X)U\right),$$

where $F : E \to E$, and $X = (\zeta, U, H)$. The integrals are defined as Riemann integrals of continuous functions on the Banach space E . For all the above, we know $X = (\zeta, U, H) \to f'(U), X = (\zeta, U, H) \to G(U), X = (\zeta, U, H) \to P(X), X = (\zeta, U, H) \to U$ are all Lipschitz from B_M to V . Then, using Lemma 2.2, we can check that each component of $F(X)$ is a product of functions that satisfy one of the assumptions of Lemma 2.2, we obtain that $F(X)$ is Lipschitz on B_M . Thus, F is Lipschitz on any bounded set of E . Since E is a Banach space, we can use the standard contraction argument and know the existence of short-time solutions. ■

Next we will proof the existence of global solutions of (2.18). We will find a particular class of initial data that matches E , see Definition 2.2 In particular, we will only consider initial data that belongs to

$$\Omega = \left\{(y, U, H) | (y, U, H) \in E \cap \left(W^{1,\infty}(R) \times W^{1,\infty}(R) \times W^{1,\infty}(R)\right)\right\}$$

Lemma 2.3 $y_\xi H_\xi = y_\xi^2 U^2 + U_\xi^2$ *almost everywhere.*

Proof. In this equality, both of the two sides differenting with respect to t and using (2.19), we get

$$\begin{aligned}
&(y_\xi H_\xi)_t = y_{\xi t} H_\xi + y_\xi H_{\xi t} \\
&= f''(U)U_\xi H_\xi + y_\xi(-2QUy_\xi + (2g(U) + f''(U)U^2 + 2f''(U)U - 2P)U_\xi) \\
&= f''(U)U_\xi H_\xi + f''(U)U^2 U_\xi y_\xi + 2f''(U)UU_\xi y_\xi + 2g(U)U_\xi y_\xi - 2PU_\xi y_\xi - 2QUy_\xi^2
\end{aligned}$$

$$\begin{aligned}
&(y_\xi^2 U^2 + U_\xi^2)_t = 2y_\xi y_{\xi t} U^2 + 2y_\xi^2 U U_t + 2U_\xi U_{\xi t} \\
&= 2f''(U)y_\xi U_\xi U^2 + 2y_\xi^2 U(-Q) + 2U_\xi \left(\tfrac{1}{2}f''(U)H_\xi - Py_\xi - \tfrac{1}{2}f''(U)U^2 y_\xi + f''(U)Uy_\xi + g(U)y_\xi\right) \\
&= f''(U)H_\xi U_\xi + f''(U)U_\xi U^2 y_\xi - 2y_\xi^2 UQ + 2f''(U)Uy_\xi U_\xi - 2PU_\xi y_\xi + 2g(U)U_\xi y_\xi
\end{aligned}$$

so $(y_\xi H_\xi)_t = (y_\xi^2 U^2 + U_\xi^2)_t$, and $y_\xi H_\xi(0) = (y_\xi^2 U^2 + U_\xi^2)(0)$, then $y_\xi H_\xi = y_\xi^2 U^2 + U_\xi^2$ almost everywhere. ■

From the above, we know $\bar{y}_\xi \bar{H}_\xi = \bar{y}_\xi^2 \bar{U}^2 + \bar{U}_\xi^2$, and later we will use it to prove $\bar{y}, \bar{H}, \bar{U}$ belong to $W^{1,\infty}, W^{1,\infty}, W^{1,\infty}$. And if $\|\bar{\zeta}_\xi\| + \|\bar{U}_\xi\| + \|\bar{H}_\xi\| < \infty$ we can prove the solutions exist in $[0,T]$, for any time $t \in [0,T]$, see [21, Lemma 2.3].

Definition 2.2 $\bar{\xi} = \int\limits_{-\infty}^{\bar{y}(\xi)} (\bar{u}^2 + \bar{u}_x^2)dx + \bar{y}(\xi)$, $\bar{U} = \bar{u} \circ \bar{y}$, $\bar{H} = \int\limits_{-\infty}^{\bar{y}} (\bar{u}^2 + \bar{u}_x^2)dx$, $y(\xi) = \zeta(\xi) + \xi$, $\varepsilon = \left\{(f,g)|(f,g) \in W_{loc}^{2,\infty}(\mathbb{R}) \times W_{loc}^{1,\infty}(\mathbb{R})\right\}$, *see [25].*

It is easy to prove $H_\xi \geqslant 0$, H_ξ is an increasing function with respect to ξ . We have $\lim\limits_{\xi \to \pm\infty} U(t,\xi) = 0$, and $H(t,\xi) = H(0,\xi) + \int\limits_0^t [G(U) - 2PU](\tau,\xi)d\tau$. Hence we can prove $H(t,\pm\infty) = H(0,\pm\infty)$, so $\lim\limits_{\xi \to \pm\infty} H(t,\xi)$ exists and is independent of time. Let's define $\sup\limits_{t\in[0,T]} \|H(t,\cdot)\|_{L^\infty(R)} = \|\bar{H}\|_{L^\infty(R)} = h$.

Lemma 2.4 *Given $\bar{u} \in H^1(\mathbb{R})$, the initial date $(\bar{y}, \bar{U}, \bar{H})$ belongs to Ω .*

Proof. $\bar{\zeta}_\xi = -(\bar{u}^2 + \bar{u}_x^2) \circ \bar{y}\bar{y}_\xi$, $\bar{y}_\xi = 1 + \bar{\zeta}_\xi$ we get $\bar{\zeta}_\xi(\xi) = -\frac{\bar{u}^2+\bar{u}_x^2}{1+\bar{u}^2+\bar{u}_x^2} \circ \bar{y}(\xi)$, so $\bar{\zeta}_\xi$ is bounded almost everywhere and $\bar{\zeta}$ belongs to $W^{1,\infty}(\mathbb{R})$. ($\bar{y}_\xi = \frac{1}{1+\bar{u}^2+\bar{u}_x^2} \circ \bar{y}$, and $0 < \bar{y}_\xi < 1$ almost everywhere, so $\bar{y}$ belongs to $W^{1,\infty}(\mathbb{R})$.)

$\bar{H} = -\bar{\zeta}$ and $\bar{H}$ belongs to $W^{1,\infty}(\mathbb{R})$. From $\bar{y}_\xi \bar{H}_\xi = \bar{y}_\xi^2 \bar{U}^2 + \bar{U}_\xi^2$, we know $\bar{U}_\xi^2 \leqslant \bar{y}_\xi \bar{H}_\xi$, so $\bar{U}$ belongs to $W^{1,\infty}(\mathbb{R})$.

For every smooth function ϕ using the change of variable $x = y(\xi)$, and

$$\int\limits_{\mathbb{R}} u\phi dx = \int\limits_{\mathbb{R}} U(\phi \circ y)y_\xi d\xi = \int\limits_{\mathbb{R}} U\sqrt{y_\xi}(\phi \circ y)\sqrt{y_\xi}d\xi,$$

$\left|\int\limits_{\mathbb{R}} u\phi dx\right| \leqslant \|\phi\|_{L^2(\mathbb{R})} \sqrt{\int\limits_{\mathbb{R}} U^2 y_\xi d\xi} \leqslant \sqrt{H(\infty)}\,\|\phi\|_{L^2(\mathbb{R})}$ Cauchy-Schwarz inequality

For $\|u\|_{L^2(\mathbb{R})} \leqslant \sqrt{H(t,\infty)} = \sqrt{H(0,\infty)} = \|\bar{u}\|_{H^1(\mathbb{R})}$ see [26],

$$\int\limits_{\mathbb{R}} u\phi_x(x)dx = \int\limits_{\mathbb{R}} U(\xi)\phi_x(y(\xi))y_\xi(\xi)d\xi = -\int\limits_{\mathbb{R}} U_\xi(\xi)(\phi \circ y)(\xi)d\xi,$$

let $B = \{\xi \in \mathbb{R} | y_\xi(\xi) > 0\}$,

$$\left|\int\limits_{\mathbb{R}} u\phi_x dx\right| = \left|\int\limits_B \frac{U_\xi}{\sqrt{y_\xi}}(\phi \circ y)\sqrt{y_\xi}d\xi\right| \leqslant \sqrt{\int\limits_B \frac{U_\xi^2}{y_\xi}d\xi}\sqrt{\int\limits_B (\phi \circ y)^2 y_\xi d\xi} \leqslant \sqrt{H(\infty)}\,\|\phi\|_{L^2(\mathbb{R})},$$

so $u_x \in L^2(\mathbb{R}), \|u_x\|_{L^2(\mathbb{R})} \leqslant \|\bar{u}\|_{H^1(\mathbb{R})} = \sqrt{h}, U_\xi = u_x \circ y y_\xi$, therefore $U_\xi \leqslant \sqrt{h}$, and $U \in W^{1,\infty}(\mathbb{R})$.

From all the above, we know $(\bar{\zeta}, \bar{U}, \bar{H}) \in \Omega$. ∎

Theorem 2.3 *The system of (2.18) has a unique global solution $X(t) = (y(t), U(t), H(t))$ in $C^1(\mathbb{R}_+, E)$ with the initial date $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{H})$.*

Proof. Next we will prove $\sup\limits_{t\in[0,T]} \|X(t)\|_E < \infty$,

$$U^2(\xi) = 2\int\limits_{-\infty}^{\xi} U(\eta)U_\xi(\eta)d\eta = 2\int\limits_{\{\eta\leqslant\xi|y_\xi(\eta)>0\}}^{\xi} U(\eta)U_\xi(\eta)d\eta$$

$$|U(\xi)U_\xi(\xi)| = \left|\sqrt{y_\xi}U_\xi(\xi)\frac{U_\xi(\xi)}{\sqrt{y_\xi}}\right| \leqslant \frac{1}{2}\left(U(\xi)^2 y_\xi(\xi) + \frac{U_\xi^2(\xi)}{y_\xi(\xi)}\right) = \frac{1}{2}H_\xi(\xi),$$

We get $U^2(\xi) \leqslant H(\xi)$, $U(t,\xi) \in I := [-\sqrt{h}, \sqrt{h}]$ for all the $t \in [0,T], \xi \in \mathbb{R}$, $\sup\limits_{t\in[0,T]} \|U(t,\cdot)\|_{L^\infty(\mathbb{R})} < \infty$.

We get $\kappa = \|f\|_{W^{2,\infty}(I)} + \|g\|_{W^{1,\infty}(I)}$ see [27], $|\zeta(t,\xi)| \leq |\zeta(0,\xi)| + \kappa T$, so $\sup\limits_{t\in[0,T]} \|\zeta(t,\cdot)\|_{L^\infty(\mathbb{R})}$ is bounded.

From (2.16), we know $\|g(U)\|_{L^2(R)} \leqslant \|g\|_{W^{1,\infty}(R)} \|U\|_{L^2(R)} \leqslant k\sqrt{h}$,

$$
\begin{aligned}
|Q(t,\xi)| &\leqslant \tfrac{3kh+4k\sqrt{h}}{4} \int_{\mathbb{R}} e^{-|y(\xi)-y(\eta)|} y_\xi(\eta) d\eta \\
&= \tfrac{3kh+4k\sqrt{h}}{4} \int_{\mathbb{R}} e^{-|y(\xi)-x|} dx = \tfrac{3kh}{2} + 2k\sqrt{h}.
\end{aligned}
$$

So Q (P) is bounded by a constant that depends only on κ, h .

Let

$$
C_2 = \sup_{t\in[0,T]} \left\{ \|U(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|H(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|\zeta(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|P(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|Q(t,\cdot)\|_{L^\infty(\mathbb{R})} \right\}
$$

(C_2 is finite and depends only on $\|\bar{X}\|_E, T, \kappa$)

The same bounds hold for Q, P ,

$$
Z(t) = \|U(t,\cdot)\|_{L^2(\mathbb{R})} + \|U_\xi(t,\cdot)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\cdot)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\cdot)\|_{L^2(\mathbb{R})}
$$

From all the above, we get

$Z(t) \leqslant Z(0) + C \int_0^t Z(\tau) d\tau$, by Grownwall's lemma, $\sup\limits_{t\in[0,T]} Z(t) < \infty$, so the solutions exist globally in time, and we finish our proof. ■

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