Nonlocal Boundary Value Problem of a Fractional-Order Functional Differential Equation

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Abstract: The topic of fractional calculus, (integration and differentiation of fractional-order) is a one of the singular integral and integro-differential operators (see [1], [5]-[8], [10]-[11],[14]-[16] and [18] and the references therein). In this work, we prove some local and global existence theorems for a nonlocal nonlinear boundary value problem of a fractional-order functional differential equation.

Keywords: fractional calculus; nonlocal nonlinear boundary value problem of a fractional-order functional differential equation

1 Introduction

The three-point boundary value problem has been studied by many authors (see [9, 12] and [19, 20]) for instance. In [13], they study the existence of positive solutions to the three-point boundary value problem

\[
\begin{cases}
u'' + a(t) f(u) = 0, & t \in (0,1), \\
u(0) = 0, & 0 < \eta < 1, \quad 0 < \alpha < \frac{1}{\eta}
\end{cases}
\]

They proved the existence of at least one positive solution if \( f \) is either superlinear or sublinear by applying the fixed point theorems in cones. Also, in [17], the author concerned with determining values for \( \lambda \) so that the three-point nonlinear second order boundary value problem

\[
\begin{cases}
u''(t) + \lambda a(t) f(u(t)) = 0, & t \in (0,1), \\
u(0) = 0, & 0 < \eta < 1, \quad 0 < \alpha < \frac{1}{\eta}
\end{cases}
\]

has positive solutions.

Now let \( \beta \in (1,2) \) and \( \gamma \in (0,1] \), we deal here with the nonlocal nonlinear boundary value problem of a fractional-order functional differential equation

\[
\begin{cases}D^\beta u(t) + f(t, u(\phi(t))) = 0, & t \in (0,1), \\
I^\gamma u(t)|_{t=0} = 0, & 0 < \eta < 1, \quad 0 < \alpha \eta^{\beta-1} < 1.
\end{cases}
\]  

We investigate the behavior of solutions for problem (1) with certain nonlinearities, using the equivalence of the problem with the corresponding integral equation, we prove the existence of \( L_1 \)-solution such that the function \( f \) satisfies the Caratheodory conditions and the growth condition.

2 Preliminaries

Let \( L_1(I) \) be the class of Lebesgue integrable functions on the interval \( I = [a,b] \), \( 0 \leq a < b < \infty \) and \( \Gamma(\cdot) \) be the gamma function.

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Recall that the operator $T$ is compact if it is continuous and maps bounded sets into relatively compact ones. The set of all compact operators from the subspace $U \subset X$ into the Banach space $X$ is denoted by $C(U, X)$. Moreover, we set $B_r = \{ u \in L^1(I) : \| u \| < r, r > 0 \}$.

**Definition 2.1** The fractional integral of the function $f(\cdot) \in L^1(I)$ of order $\beta \in R^+$ is defined by (see [14]-[16] and [18])

$$I^\beta_a f(t) = \int_a^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)} f(s) \; ds.$$  

**Definition 2.2** The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0, 1)$ is defined as (see [14]-[16] and [18])

$$D^\alpha_a f(t) = \frac{d}{dt} I^{1 - \alpha}_a f(t), \quad t \in [a, b].$$

Now the following theorem (some properties of the fractional-order integration) can be easily proved.

**Theorem 2.1** Let $\beta, \gamma \in R^+$ and $\alpha \in (0, 1]$. Then we have:

(i) $I^\beta _a : L_1 \rightarrow L_1$, and if $f(t) \in L_1$, then $I^\gamma _a I^\beta _a f(t) = I^{\gamma + \beta} _a f(t)$.

(ii) $\lim _{\beta \rightarrow n} I^\beta _a f(t) = I^n_a f(t)$, $n = 1, 2, 3, \cdots$ uniformly.

Now, let us recall some results which will be needed in the sequel.

**Theorem 2.2 (Rothe Fixed Point Theorem)** [3]
Let $U$ be an open and bounded subset of a Banach space $E$, let $T \in C(\bar{U}, E)$. Then $T$ has a fixed point if the following condition holds

$$T(\partial U) \subseteq \bar{U}.$$

**Theorem 2.3 (Nonlinear alternative of Laray-Schauder type)** [3]
Let $U$ be an open subset of a convex set $D$ in a Banach space $E$. Assume $0 \in U$ and $T \in C(\bar{U}, E)$. Then either

(A1) $T$ has a fixed point in $\bar{U}$, or

(A2) there exists $\ell \in (0, 1)$ and $x \in \partial U$ such that $x = \ell Tw$.

**Theorem 2.4 (Kolmogorov compactness criterion)** [4]
Let $\Omega \subseteq L^p (0, 1)$, $1 \leq p < \infty$. If

(i) $\Omega$ is bounded in $L^p (0, 1)$, and

(ii) $x_h \rightarrow x$ as $h \rightarrow 0$ uniformly with respect to $x \in \Omega$, then $\Omega$ is relatively compact in $L^p (0, 1)$, where

$$x_h(t) = \frac{1}{h} \int_t^{t + h} x(s) \; ds.$$

### 3 Main results

We begin this section by proving the equivalence of the problem (1) with the functional integral equation:

$$u(t) = - I^\beta f(t, u(\phi(t))) - \frac{\alpha t^{\beta - 1}}{1 - \alpha} \int_0^\eta \frac{(\eta - s)^{\beta - 1}}{\Gamma(\beta)} f(s, u(\phi(s))) \; ds$$

$$+ \frac{t^{\beta - 1}}{1 - \alpha} \int_0^1 \frac{(1 - s)^{\beta - 1}}{\Gamma(\beta)} f(s, u(\phi(s))) \; ds. \quad (2)$$

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Lemma 3.1 The nonlocal nonlinear boundary value problem
\[ D^\beta u(t) + f(t, u(\phi(t))) = 0, \quad \beta \in (1, 2), \quad t \in (0, 1), \]
\[ I^\gamma u(t)|_{t=0} = 0, \quad \gamma \in (0, 1], \quad \alpha u(\eta) = u(1), \quad 0 < \eta < 1, \quad 0 < \alpha \eta^{\beta-1} < 1 \]
is equivalent to the fractional-order functional integral equation (2)

\[ u(t) = -I^\beta f(t, u(\phi(t))) + C_1 t^{\beta-1} + C_2 t^{\beta-2}. \]

By (4), we get \( C_2 = 0 \) and
\[ C_1 = \frac{-\alpha}{1 - \alpha \eta^{\beta-1}} \int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} f(s, u(\phi(s))) ds + \frac{1}{1 - \alpha \eta^{\beta-1}} \int_0^1 \frac{(1 - s)^{\beta-1}}{\Gamma(\beta)} f(s, u(\phi(s))) ds. \]

Therefore, the solution of problem (3), (4) is given by the formula (2).
Conversely, let \( u(t) \) be a solution of (2) and operating on both sides of it by \( I^{2-\beta} \), we get
\[ I^{2-\beta} u(t) = -I^2 f(t, u(\phi(t))) - t C_3 + t C_4. \]

Differentiating the last relation two times we obtain (3), also it is easy to check that conditions (4) are satisfied. The proof is complete.

To facilitate our discussion, let us first state the following assumptions:

(i) \( f : (0, 1) \times R \to R^+ \) be a function with the following properties:

1. for each \( t \in (0, 1) \), \( f(t, \cdot) \) is continuous,
2. for each \( u \in R \), \( f(\cdot, u) \) is measurable,
3. there exist two real functions \( a(t), b(t) \) such that
\[ f(t, u) \leq a(t) + b(t) | u |, \quad \text{for each } t \in (0, 1), \quad u \in R, \]

where \( a(\cdot) \in L_1(0, 1) \) and \( b(\cdot) \) is measurable and bounded.

(ii) \( \phi : (0, 1) \to (0, 1) \) is nondecreasing and there is a constant \( M > 0 \) such that \( \phi' \geq M \) a.e. on \( (0, 1) \).

Also, define the operator \( T \) as
\[ Tu(t) = -I^\beta f(t, u(\phi(t))) - \frac{\alpha}{1 - \alpha \eta^{\beta-1}} \int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} f(s, u(\phi(s))) ds + \frac{t^{\beta-1}}{1 - \alpha \eta^{\beta-1}} \int_0^1 \frac{(1 - s)^{\beta-1}}{\Gamma(\beta)} f(s, u(\phi(s))) ds. \]

To solve equation (2) it is necessary to find a fixed point of the operator \( T \).

For the local existence of a solution we have the following theorem:

**Theorem 3.2** Let the assumptions (i) and (ii) are satisfied.

\[ \text{If } \sup |b(t)| < M K, \quad (5) \]

where \( K = (1 - \alpha \eta^{\beta-1}) \frac{1}{\Gamma(1 + \beta)} \), then the nonlocal boundary value problem (1) has a solution \( u \in B_r \), where
\[ r \leq \frac{1}{1 - \frac{1}{MK}} \sup |b(t)|. \]

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Proof: Let $u$ be an arbitrary element in $B_r$. Then from the assumptions (i) - (ii), we have

$$\|Tu(t)\| \leq \frac{r^{\beta-1}}{1 - \alpha \eta^{\beta-1}} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s, u(\phi(s))) \, ds.$$  

Then

$$\|Tu\| = \int_0^1 |Tu(t)| \, dt$$

$$\leq \int_0^1 \left( \frac{r^{\beta-1}}{1 - \alpha \eta^{\beta-1}} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s, u(\phi(s))) \, ds \right) \, dt$$

$$\leq \frac{1}{1 - \alpha \eta^{\beta-1}} \int_0^1 \int_0^1 \frac{t^{\beta-1}}{\Gamma(\beta)} \, dt |f(s, u(\phi(s)))| \, ds$$

$$= \frac{1}{1 - \alpha \eta^{\beta-1}} \int_0^1 \left( \frac{t^{\beta}}{\Gamma(1 + \beta)} \right)_0^1 |f(s, u(\phi(s)))| \, ds$$

$$= \frac{1}{K} \int_0^1 |f(s, u(\phi(s)))| \, ds$$

$$\leq \frac{1}{K} \int_0^1 (|a(s)| + |b(s)| |u(\phi(s))|) \, ds$$

$$\leq \frac{1}{K} \left( \|a\| + \sup b(t) \int_0^1 |u(\phi(s))| \, ds \right)$$

$$\leq \frac{1}{K} \left( \|a\| + \sup b(t) \cdot \frac{1}{M} \int_0^1 |u(\phi(s))| \, ds \right)$$

$$= \frac{1}{K} \left( \|a\| + \sup b(t) \cdot \frac{1}{M} \int_\phi^{\phi(t)} |u(x)| \, dx \right)$$

$$\leq \frac{1}{K} \left( \|a\| + \sup b(t) \cdot \frac{1}{M} \|u\| \right).$$

The last estimate shows that the operator $T$ maps $L_1$ into itself. Now, let $u \in \partial B_r$, that is, $\|u\| = r$, then the last inequality implies

$$\|Tu\| \leq \frac{1}{K} \left( \|a\| + \sup |b(t)| \cdot \frac{1}{M} r \right).$$

Then $T(\partial B_r) \subseteq B_r$ (closure of $B_r$) if

$$\|Tu\| \leq \frac{1}{K} \left( \|a\| + \sup |b(t)| \cdot \frac{1}{M} r \right) \leq r,$$

which implies that

$$\frac{1}{K} \left( \|a\| + \sup |b(t)| \cdot \frac{1}{M} r \right) \leq r.$$

Therefore

$$r \leq \frac{\frac{1}{K} \|a\|}{1 - \frac{1}{MK} \sup |b(t)|}.$$  

Using inequality (5) we deduce that $r > 0$. Moreover, we have

$$\|f\| = \int_0^1 |f(s, u(\phi(s)))| \, ds$$

$$\leq \int_0^1 (|a(s)| + |b(s)| |u(\phi(s))|) \, ds$$

$$\leq \|a\| + \sup |b(t)| \cdot \frac{1}{M} \|u\|.$$  

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This estimation shows that $f$ in $L_1(0, 1)$.

Further, $f$ is continuous in $u$ (assumption 1) and $I^\alpha$ maps $L_1(0, 1)$ continuously into itself, $I^\alpha f(t, u(\phi(t)))$ is continuous in $u$. Since $u$ is an arbitrary element in $B_r$, $T$ maps $B_r$ continuously into $L_1(0, 1)$.

Now, we will show that $T$ is compact, to achieve this goal we will apply Theorem 2.4. So, let $\Omega$ be a bounded subset of $B_r$. Then $T(\Omega)$ is bounded in $L_1(0, 1)$, i.e. condition (i) of Theorem 2.4 is satisfied. It remains to show that $(Tu)_h \to Tu$ in $L_1(0, 1)$ as $h \to 0$, uniformly with respect to $Tu \in T \Omega$. We have the following estimation:

$$\| (Tu)_h - Tu \| = \int_0^1 \left| (Tu)_h(t) - (Tu)(t) \right| \, dt$$

$$= \int_0^1 \frac{1}{h} \int_t^{t+h} (Tu)(s) \, ds - (Tu)(t) \, dt$$

$$\leq \int_0^1 \left( \frac{1}{h} \int_t^{t+h} |(Tu)(s) - (Tu)(t)| \, ds \right) \, dt$$

$$\leq \int_0^1 \frac{1}{h} \int_t^{t+h} \left| - I^\beta f(s, u(\phi(s))) + I^\beta f(t, u(\phi(t))) \right| \, ds \, dt$$

$$+ \int_0^1 \frac{1}{h} \int_t^{t+h} \left| - K_1 s^{\beta-1} + K_1 t^{\beta-1} \right| \, ds \, dt$$

$$+ \int_0^1 \frac{1}{h} \int_t^{t+h} \left| K_2 s^{\beta-1} - K_2 t^{\beta-1} \right| \, ds \, dt,$$

where

$$K_1 = \frac{\alpha}{1 - \alpha \eta^{\beta-1}} \int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} f(s, u(\phi(s))) \, ds$$

and

$$K_2 = \frac{1}{1 - \alpha \eta^{\beta-1}} \int_0^1 \frac{(1 - s)^{\beta-1}}{\Gamma(\beta)} f(s, u(\phi(s))) \, ds.$$

Since $f \in L_1(0, 1)$ we get that $I^\beta f(.) \in L_1(0, 1)$. Moreover $\eta^{\beta-1} \in L_1(0, 1)$. So, we have (see [21])

$$\frac{1}{h} \int_t^{t+h} \left| - I^\beta f(s, u(\phi(s))) + I^\beta f(t, u(\phi(t))) \right| \, ds \to 0,$$

$$\frac{1}{h} \int_t^{t+h} \left| - K_1 s^{\beta-1} + K_1 t^{\beta-1} \right| \, ds \to 0$$

and

$$\frac{1}{h} \int_t^{t+h} \left| K_2 s^{\beta-1} - K_2 t^{\beta-1} \right| \, ds \to 0$$

for a.e. $t \in (0, 1)$. Therefore, by Theorem 2.4, we have that $T(\Omega)$ is relatively compact, that is, $T$ is a compact operator.

Therefore, Theorem 2.2 with $U = B_r$ and $E = L_1(0, 1)$ implies that $T$ has a fixed point. This complete the proof.  

Now for more global solution of the nonlocal boundary value problem (1), consider the following assumption:

(iii) Assume that every solution $u(.) \in L_1(0, 1)$ to the equation

$$u(t) = \ell (- I^\beta f(t, u(\phi(t))) - \frac{\alpha}{1 - \alpha \eta^{\beta-1}} \int_0^\eta \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} f(s, u(\phi(s))) \, ds$$

$$+ \frac{t^{\beta-1}}{1 - \alpha \eta^{\beta-1}} \int_0^1 \frac{(1 - s)^{\beta-1}}{\Gamma(\beta)} f(s, u(\phi(s))) \, ds)$$

satisfies $\|u\| \neq r$ ($r$ is arbitrary but fixed).
Theorem 3.3 Let the conditions (i) - (iii) be satisfied, then the nonlocal boundary value problem (1) has at least one solution \( u \in L_1(0,1) \).

Proof: Let \( u \) be an arbitrary element in the open set \( B_r = \{ u : ||u|| < r, r > 0 \} \). Then from the assumptions (i) - (ii), we have

\[
||Tu|| \leq \frac{1}{K} \left( ||a|| + \sup |b(t)| \frac{1}{M} ||u|| \right).
\]

The above inequality means that the operator \( T \) maps \( B_r \) into \( L_1 \). Moreover, we have

\[
||f|| \leq ||a|| + \sup |b(t)| \frac{1}{M} ||u||.
\]

This estimation shows that \( f \) in \( L_1(0,1) \).

As a consequence of Theorem 3.2 we get that \( T \) maps \( B_r \) continuously into \( L_1(0,1) \) and \( T \) is compact. Set \( U = B_r \) and \( D = E = L_1(0,1) \), then in the view of assumption (iii) the condition \( A_2 \) of Theorem 2.3 does not hold. Therefore, Theorem 2.3 implies that \( T \) has a fixed point. This complete the proof.

References


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