New Compactons in Nonlinear Atomic Chain Equations with first-and second-neighbour Interactions

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Abstract: Multiple compactons in a nonlinear atomic chain equations are studied. Atoms in the chain are interacted through first-and-second interactions. Nonlinearity in the evolution equation is set to be up to cubic polynomial. The spatial and temporal dependence of the solutions are given by separating method. Multiple $N$–site compactons are obtained.

Keywords: multiple compacton; discrete system; lattice model; chain equation

1 Introduction

The concept of compactons: solitons with compact support, or strict localization of solitary waves appeared recently in many branches of science [1]. The investigation of discrete models helps us to overcome difficulties of numerical simulations connected with the application of finite difference methods [2]. Therefore the problem of the appearance of compactons in discrete systems is of great interest because discrete systems such as the Toda chain, discrete spin lattice models, etc. arise in the study of many problems of condensed matter physics[3].

In continuous systems, great interest has been shown in compactons [4]-[11]. Fan and Tian proved the existence question of solitary wave of mKdV-KS equation [7]. Tian and Yin obtain multi-compaction solutions, floating compacton solutions, etc. for many PDEs [4]-[6]. In the lattice case, dynamics in nonlinear lattices display a great diversity of excitations. R Han ,et al [12] obtained exact traveling kink-like wave solution to nonlinear atomic chain equation. Kevrekids and Konotop [13] found a suitable model possessing compacton solutions and gave exact discrete solutions with compact support to the discrete nonlinear Klein-Gordon models. They presented the $N$–site compacton for periodic lattice excitations which are strictly localized on $N$–sites and are zero everywhere else.

In this article, we study a monatomic chain of interacting particles including first-and-second-neighbor interactions. We obtain lattice compactons, which are the discrete analog of solutions known to exist in special classes of PDEs. We find some new compactons for discrete systems due to the second neighbor interactions.

The rest of this paper is organized as follows. In section 2, we give the chain model possessing compactons by analytical considerations. Then in section 3 we present the exact new compactons and some numerical results. In section 4 we give the linear stability of multiple compactons.
The equations of motion for the finite chain are
\[ \ddot{y}_n = \Delta_1 y_n + \epsilon \Delta_2 y_n + f_1(u_{n-1}, u_n, u_{n+1}) + f_2(u_{n-2}, u_n, u_{n+2}) \]  
\[ \text{where } n = 1, 2, ..., N, \text{ and } y_n(t) \text{ denotes the longitudinal displacement of the } n \text{th atom with unit mass. And} \]
\[ \Delta_1 y_n = y_{n+1} + y_{n-1} - 2y_n, \]
\[ \Delta_2 y_n = y_{n+2} + y_{n-2} - 2y_n, \]
are the discrete Laplacian with unit spacing, and } \epsilon \text{ is a small scaling parameter.}

The nonlinearity is assumed to have the following property
\[ f_1(s y_{n-1}, s y_n, s y_{n+1}) = \sum_{k=1}^{M} g_k(s y_{n-1}, s y_n, s y_{n+1}) = \sum_{k=1}^{M} s^k g_k(y_{n-1}, y_n, y_{n+1}), \]
\[ f_2(s y_{n-2}, s y_n, s y_{n+2}) = \sum_{k=1}^{M_2} h_k(s y_{n-2}, s y_n, s y_{n+2}) = \sum_{k=1}^{M_2} s^k h_k(y_{n-2}, y_n, y_{n+2}), \]
where \( s \) is any function of time and \( M_2 \leq M \) because the first neighbor interaction is considered to be stronger than the second. For simplicity in the following we will set \( M_2 = M = 3 \).

As in continuous systems compacton solutions are zero outside a finite domain of space variable. In the lattice applications, \( N \)– site compactons were introduced in [13] which means solutions strictly localized on \( N \)– site and are zero everywhere else. We introduce multiple compacton solutions for the lattice model, which may be zero inside the \( N \)– site but are strictly zero outside the \( N \)– sites, i.e. for solutions periodic in time and strictly localized on \( N \) neighboring lattice sites such that
\[ y_n(t) = y_n(t + T); \]
and there exist at least two \( n_1, n_2 \), such that
\[ y_{n_1}(t) \neq 0, \quad \text{if } n_1 = n_0, ..., n_0 + N - 1; \]
\[ y_{n_2}(t) = 0, \quad \text{if } n < n_0 \text{ and } n_0 \geq N. \]

Notice that our concept of multiple compacton means some \( y_n \) (not all \( y_n \)) can be zero when \( n_0 \leq n \leq n_0 + N - 1 \) while in[13] each \( y_n \), \( n_0 \leq n \leq n_0 + N - 1 \) must not be zero.

Now we look for the form of multiple compactons for Eq.(1). Separating the temporal and spatial dependencies, we assume
\[ y_n(t) = s(t)v_n \]
where \( s(t + T) = s(t) \) is function with period \( T \). Substitute (2) into (1) we have
\[ s = \sum_{k=1}^{M} F_k(v_{n-2}, v_{n-1}, v_n, v_{n+1}, v_{n+2})s^k(t), \]
where
\[ F_1(v_{n-2}, v_{n-1}, v_n, v_{n+1}, v_{n+2}) = \frac{\Delta_1 v_n + \epsilon \Delta_2 v_n}{v_n} + \frac{g_1(v_{n-1}, v_n, v_{n+1}) + h_1(v_{n-2}, v_n, v_{n+2})}{v_n}, \]
\[ F_k(v_{n-2}, v_{n-1}, v_n, v_{n+1}, v_{n+2}) = \frac{g_k(v_{n-1}, v_n, v_{n+1}) + h_k(v_{n-2}, v_n, v_{n+2})}{v_n}, k = 2, 3, ..., M. \]
As is evident, the consistency of the solution (2) with the system (3) implies that $F_k = C_k$, where $C_k$ is the constant independent on $n$. Then Equ.(3) is reduced to

$$\ddot{s} + \sum_{k=1}^{M} C_k s^k = 0,$$

and thus

$$\int_{0}^{s} \frac{ds}{\sqrt{E - P_M(s)}} = \sqrt{2}(t - t_0), P_M(s) = \sum_{k=1}^{M} \frac{C_k}{k+1} s^{k+1}, \quad (4)$$

where $t_0$ and $E$ are constants. We will be interested in the case where $C_1 \neq 0$ and thus without restriction of generality in what follows we set $C_1 = 1$ and $t_0 = 0$. This can be done by renormalizing time. Then the temporal dependence of the compacton amplitude is

$$s(t) = \frac{\sqrt{2E}}{\omega} sn(\omega(t - t_0)|m), \quad (5)$$

where $sn(|m)$ is the Jacobi elliptic sine function of modulus $m$,

$$m = \frac{1 - \sqrt{1 - 4C_3E}}{1 + \sqrt{1 - 4C_3E}},$$

$$\omega^2 = \frac{1 - \sqrt{1 - 4C_3E}}{2}$$

and it is assumed that $0 < 4C_3E < 1$. Then the period of the solution of Equ.(1) is

$$T = \frac{4K(m)}{\omega}$$

where $K(m)$ is the complete elliptic integral of the first kind.

Notice that the equation $F_1 = -1$ implies the absence of linear dispersion, which is a common feature of model supporting compacton solution. Secondly, because one has to require structural stability of a compacton and coupling between the first and the second sites, we have turn to cubic nonlinearity. Taken symmetric with respect to the first neighbors and the second neighbors, excluded the next nearest neighbor interaction between sites $n-1$ and $n+1$ and that between sites $n-2$ and $n-2$, the cubic term is taken as

$$A_1 v_n^2 (v_{n-1} + v_{n+1}) + A_2 v_n^2 (v_{n-2} + v_{n+2}) + A_3 v_n^3.$$\n
Thus the spacial profile is defined by

$$A_1 v_n (v_{n-1} + v_{n+1}) + A_2 v_n (v_{n-2} + v_{n+2}) + A_3 v_n^2 = -C_3. \quad (6)$$

The evolution equation (1) now can be written in the form

$$\ddot{y}_n = -y_n - \epsilon y_n + A_1 y_n^2 (y_{n-1} + y_{n+1}) + A_2 y_n^2 (y_{n-2} + y_{n+2}) + A_3 y_n^3. \quad (7)$$

### 3 New multiple compactons

Based on Eq.(5) and Eq. (6), we can get some new solutions to Equ. 7. The cubic on-site nonlinearity in Eq. (7) arises from a quadric substrate potential. Such potentials (both soft and hard) [13] are commonly used in a variety of physical setting ranging from the study of phase transitions to that of simple polymeric chains, particle interactions in nuclear physics or quantum field theory. The nearest neighbor coupling term can be viewed as a deformation of intersite interactions due to an on-site nonlinearity. We restrict our discussion to $A_2 \neq 0$. If $A_2 = 0$ one can refer to [13].

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3.1 case for $A_3 = 0$

This means there is only Ablowitz-Ladik (AL) nonlinearity. If $A_2 = 0$, as can be seen in [13], there exists only 2-site compactons. But once there are second neighbor interactions, more interesting phenomenon appears. More choices of spatial profiles can be made since there are two free variables in Equ.(6). This means we can obtain several new multiple compactons.

Solution 1: double compacton

Let the on-site and its second next particle be excited, then we have double compacton of spatial profile $(..., 0, 0, v_0, 0, v_0, ...)$, where $v_0 = \pm \sqrt{- \frac{C_3}{A_2}}$. Notice that $v_0$ can be positive or negative. For simplicity in what follows we will take the positive value. Taken $C_3 = 0.1, A_2 = 0.99$, Fig.(1) shows the spatial profile and the time evolution over one full period of the fourth particle. One can see in Fig.(1) that there are two excited sites that is why we name it double compacton.

![Figure 1: double compacton](image1)

Solution 2: triple compacton

Let the on-site and its both second neighbor atoms be excited in the same direction, then we get triple compacton of spatial profile $(..., v_0, 0, v_0, 0, v_0, ...)$, where $v_0 = \pm \sqrt{- \frac{C_3}{2A_2}}$. Fig.(2) shows the spatial profile and the time evolution over one full period of the fourth particle for $C_3 = 0.1, A_2 = -0.99$.

![Figure 2: triple compacton](image2)

Solution 3: wing compacton

Let the on-site and its both second neighbor atoms be excited in the reverse direction, then we have wing compacton of spatial profile $(..., -v_0, 0, v_0, 0, -v_0, ...)$, where $v_0 = \pm \sqrt{\frac{C_3}{2A_2}}$. Fig.(3) shows the spatial profile and the time evolution over one full period of the fourth particle for $C_3 = 0.1, A_2 = 0.99$. This is a special triple compacton.

![Figure 3: wing compacton](image3)

3.2 Case for $A_3 \neq 0, A_1 = 0$

As shown in the above subsection, double and triple compacton solutions exist when $A_3 = 0$. Now we turn to the case for $A_3 \neq 0$. 

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Figure 3: wing compacton with $C_3 = 0.1$ and $A_2 = 0.99$

Solution 4: We have double compacton of spatial profile \((..., 0, 0, 0, v_0, ..., )\), where \(v_0 = \pm \sqrt{-\frac{C_3}{A_3 + A_2}}\).

Solution 5: We have triple compacton of spatial profile \((..., v_0, 0, 0, v_0, ..., )\), where \(v_0 = \pm \sqrt{-\frac{C_3}{A_3 + 2A_2}}\).

Solution 6: We have wing compacton of spatial profile \((..., -v_0, 0, 0, -v_0, ..., )\), where \(v_0 = \pm \sqrt{-\frac{C_3}{A_3 + 2A_2}}\).

We can see that spatial profiles in this case are similar to those in above case. The only difference is the absolute value of \(v_0\). It shows that the on-site nonlinear coefficient \(A_3\) has no impact on the profiles.

3.3 Case for \(A_3 \neq 0, A_1 \neq 0\)

Now we consider the case when all the nonlinearity exist. Let the on-site atom and its all first and second neighbors be excited, we have 5-site compactons and name it crown compacton.

Solution 7: Crown compacton

We have crown compacton of spatial profile \((..., -v_0, v_0, -kv_0, v_0, -v_0, ..., )\), where

\[ k = \frac{(A_1 - A_2 \pm \sqrt{(A_1 - A_2)^2 + 4A_3^2})}{2A_3} \]

and

\[ v_0 = \pm \sqrt{-\frac{C_3}{-2A_1 + 2A_2 + kA_3}}. \]

Taken the positive sign in \(k\), we get two different spatial profiles for crown compactons. Fig.(4) shows the two crown compactons.

Figure 4: crown compacton with $C_3 = 0.1, A_2 = 0.99, A_3 = -0.6, A_1 = 2$ for the left but $A_3 = 0.6$ for the right

4 Linear stability

In this section, we briefly discuss the linear stability of the multiple compactons to the on-site atom, its Jacobian of the variational equations is

\[
\begin{bmatrix}
0 & 0 \\
-1 - \epsilon & -s(t)C_3
\end{bmatrix}
\]

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The eigenvalues of the Jacobian $r$ are $\pm \sqrt{-1 - \epsilon - C\[3\] * s(t)^2}$. As is known in the periodic solutions [11], if $r$ is larger than 1, the compacton will be instable. $r$ is dependent on both $\epsilon$ and $C\[3\]$. Let $\epsilon$ be positive small enough, then $C\[3\]$ will decide the stability of above compactons. Fig.(5) shows the graph of the eigenvalue $r$ as the function of $C\[3\]$. One can see that $r$ is real and is larger than 1 for some permissible value of the relevant parameters. So all the multiple compactons can be instable as $C\[3\]$ goes towards the negative axis. It is may be explained by the cause of solitary wave in the continue situation. Small nonlinearity can not balance the dispersive influence so compactons deforms.

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