A Loop Algebra and its Corresponding (2+1)-Dimensional Integrable Hierarchy

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Abstract: Searching for new integrable hierarchies of soliton equations has been an important and interesting topic in soliton theory. One of the efficient approaches to obtain integrable system is Tu scheme by constructing some loop algebra. In this paper, I discuss a (2+1)-dimensional integrable system by making use of Tu scheme and a new loop algebra. Especially when $\partial_x=0$, this (2+1) dimensional integrable system reduces to the (1+1)-dimensional generalized nonlinear Schrödinger hierarchy.

Keywords: Loop algebra; soliton equation; integrable hierarchy

1 Introduction

The study of integrable hierarchies has been an important and interesting topic in soliton theory. From a suitable isospectral problem by using loop algebras, we can obtain many integrable systems, such as AKNS hierarchy, KN hierarchy, Schrogeringer system and so on. But, up to now, there has been no general method to solve this "suitable isospectral problems", so it is an important and difficulty topic.

The Tu scheme [1] which was introduced by professor Tu GuiZhang is one of the efficient approaches to have obtained integrable hierarchy. We have derived many integrable systems by making use of the Tu scheme, such as AKNS system, CN system, WKI system and so on. One of the key step by making use of the Tu scheme is to choice a suitable loop algebra basing on some Lie algebra. As usual, we choice Lie algebra $A_{n-1} = \mathfrak{sl}(n, C)$, especially $n=2$. Obviously, it is help us to obtain new integrable system by constructing some new Lie algebra. The author in paper [2] generalized Lie algebra $A_n = \mathfrak{sl}(n, C)$ to Lie algebra $A^*_n = \mathfrak{gl}(n, C)$ and discussed corresponding integrable hierarchy basing on new algebra $A^*_1$ when $n=2$. The authors in paper [3] discussed a (2+1)-dimensional integrable hierarchy and its extension integrable hierarchy. Researches for the well-posedness, the exact solitary wave solutions and other PDE problems, have been widely studied[4]-[9]. In this paper I will discuss a (2+1)-dimensional integrable hierarchy by Tu scheme with a loop algebra and a (2+1)-dimensional Lax pair equation.

This paper is organized as follows: Section 1 is the introduction. In Section 2, we recall the general definition of Lie algebra and discuss zero curvature representation of a (2+1)-dimensional isospectral problem. In Section 3, we research (2+1)-dimensional hierarchy under a type of Loop algebra, it is a generalized of (1+1) dimensional generalized Schrödinger hierarchy.

2 A type of (2+1)-dimensional zero curvature equation under Lie algebra

For the sake of convenience, let us recall the general definition of Lie algebra.

Let $G$ be a linear space. If an operation $[x, y] \in G$, for arbitrary $x, y, z \in G$ and number $\alpha, \beta$ satisfies that:

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(i) skew-symmetry: \([x, y] = -[y, x]\),
(ii) Bilinear: \([\alpha x + \beta y, z] = \alpha [x, z] + \beta [y, z]\),
(iii) Jacobi Identity: \([[x, y], z] + [[y, z], x] + [[z, x], y] = 0\).

Then we call \(G\) a Lie algebra, the operation \([,]\) is known as a commutator. The most important Lie algebra reads that
\[ A_{n-1} = sl(n, C) = \{ X = (x_{ij})_{n \times n} | x_{ij} \in C, tr X = 0 \} \] (1)
where \(C\) denotes a set of complex number. The commutator \([x,y]\) in (1) is defined as
\[ [X, Y] = XY - YX. \] (2)

Denote
\[ gl(n, C) = \{ X = (x_{ij})_{n \times n} | x_{ij} \in C \}. \] (3)

For given \(M \in gl(n, C)\), \(\forall X, Y \in gl(n, C)\), we define
\[ [X, Y]_M = XMY - YMX. \] (4)

Obviously, the operation (4) meets (i)—-(iii). Hence \(gl(n, C)\) is a Lie algebra. If we take \(M\) be a unit matrix, then (4) reduced to (2), so the Lie algebra \(gl(n, C)\) is an extension of Lie algebra \(A_{n-1}\). Since \(M\) is arbitrary, such extension is various. Furthermore, we find that compatibility of a Lax pair by using \(gl(n, C)\) may give rise to zero curvature equation. In fact, we can consider a pair of (2+1)-dimensional isospetral problem
\[ \begin{cases} \varphi_y = A\varphi_x + UM\varphi \\ \varphi_t = B\varphi_x + VM\varphi \end{cases} \] (5)
with \(\lambda_t = 0, A_t = A_x = A_y = 0, B_t = B_x = B_y = 0\), where \(\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T\) and \(A, B, M, U, V \in gl(n, C)\).

From the first equation of (5), we have
\[ \varphi_{yt} = (\varphi_y)_t = A\varphi_{xt} + U_t M \varphi + UM \varphi_t \]
\[ = (AB)\varphi_{xx} + (AVM + UMB)\varphi_x + (U_t + AV_x U MV) M \varphi, \] (6)
and from the second equation of (5), we have
\[ \varphi_{ty} = (BA)\varphi_{xx} + (VMA + BUM)\varphi_x + (V_y + BU_x VMU) M \varphi. \] (7)

Hence, (5)’s compatible condition \(\varphi_{ty} = \varphi_{yt}\) is given by
\[ \begin{cases} [A, B] = 0 \\ [A, VM] = [B, U M] \end{cases} \] (8)
\[ U_t - V_y + AV_x - BU_x + [U, V]_M = 0. \]

Espaecially, if we take \(A=B=\text{unit matrix}\), then (5) is reduced to
\[ \begin{cases} \varphi_y = \varphi_x + UM \varphi \\ \varphi_t = \varphi_x + VM \varphi \end{cases} \] (9)
and its compatible condition \(\varphi_{ty} = \varphi_{yt}\) leads to the (2+1)-dimensional zero curvature equation
\[ U_t - V_y + V_x - U_x + [U, V]_M = 0. \] (10)

In this paper, we try to seek for (2+1)-dimensional integrable hierarchies by Tu scheme and the generalized zero curvature equation (10).

In what follows, for the sake of simplicity, we only consider a special Lie algebra \(gl(2, C)\), for which one case be considered respectively.

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3 A type of Loop algebra and its corresponding (2+1)-dimensional hierarchy

We consider a Lie algebra

\[ gl(2, C) \quad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  \hspace{1cm} (11)

Let

\[ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \]  \hspace{1cm} (12)

It is easy to check the following relation

\[ [e_1, e_2]_M = e_3, \quad [e_1, e_3]_M = -2e_1, \quad [e_2, e_3]_M = 2e_2. \]  \hspace{1cm} (13)

We define

\[ e_1(n) = e_1 \otimes \lambda^n = e_1 \lambda^{2n+1}, \quad e_2(n) = e_2 \otimes \lambda^n = e_2 \lambda^{2n+1}, \quad e_3(n) = e_3 \otimes \lambda^n = e_3 \lambda^{2n}. \]

Then

\[ \begin{cases} [e_1(m), e_2(n)]_M = e_3(m + n + 1) \\ [e_1(m), e_3(n)]_M = -2e_1(m + n) \\ [e_2(m), e_3(n)]_M = 2e_2(m + n) \\ \text{dege}_e_1(n) = \text{dege}_e_2(n) = 2n + 1, \text{dege}_e_3(n) = 2n. \end{cases} \]  \hspace{1cm} (14)

Thus, we construct a Loop algebra \( A^*_1 \) with the basis \( \{e_1(n), e_2(n), e_3(n) | n \in \mathbb{N} \} \), namely

\[ A^*_1 = \text{span}\{e_1(n), e_2(n), e_3(n) | n \in \mathbb{N} \}. \]

Now consider the following isospectral problem

\[ \begin{cases} \varphi_y = \varphi_x + UM\varphi, \lambda_y = 0 \\ U = \begin{pmatrix} q\lambda & \lambda^2 + s \\ -\lambda^2 - s & r\lambda \end{pmatrix} \end{cases} \]  \hspace{1cm} (15)

Then \( U \) is represented by the elements of the basis \( A^*_1 \):

\[ U = e_3(1) + qe_1(0) + re_2(0) + se_3(0). \]  \hspace{1cm} (16)

Let \( V = \sum_{m \geq 0} (a_m e_1(-m) + b_m e_2(-m) + c_m e_3(-m)) \), solving the following stationary zero curvature equation

\[ (\partial_y - \partial_x)V = V_y - V_x = [U, V]_M \]  \hspace{1cm} (17)

yields that

\[ \begin{cases} (\partial_y - \partial_x)a_m = 2a_{m+1} - 2q c_m + 2s a_m \\ (\partial_y - \partial_x)b_m = -2b_{m+1} + 2r c_m - 2s b_m \\ (\partial_y - \partial_x)c_m = q b_{m+1} - r a_{m+1} \\ a_0 = b_0 = 0, \quad c_0 = \alpha \quad (\text{non zero constant}) \end{cases} \]  \hspace{1cm} (18)

From the recurrence equation (18) we can calculate successively every term. For example, when \( n = 1 \) we have \( a_1 = \alpha q, b_1 = \alpha r \). From (18) we have

\[ c_m = -\frac{1}{2}(\partial_y - \partial_x)^{-1} q(\partial_y - \partial_x)b_m - \frac{1}{2}(\partial_y - \partial_x)^{-1} r(\partial_y - \partial_x)a_m - (\partial_y - \partial_x)^{-1}s(\partial_y - \partial_x)c_{m-1}. \]

Hence, when \( n = 2 \) we have

\[ \begin{cases} c_1 = -\frac{1}{2}(\partial_y - \partial_x)^{-1} q(\partial_y - \partial_x)(r\alpha) - \frac{1}{2}(\partial_y - \partial_x)^{-1} r(\partial_y - \partial_x)(\alpha q) = -\frac{a}{2} (qr) \\ a_2 = \frac{a}{2}[(\partial_y - \partial_x)q - 2sq - q^2r] \\ b_2 = -\frac{a}{2}[(\partial_y - \partial_x)r - 2sr - qr^2] \end{cases} \]  \hspace{1cm} (19)

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Denote
\[
V^{(n)}_+ = \sum_{m=0}^{n} [a_m e_1(n-m) + b_m e_2(n-m) + c_m e_3(n-m)],
\]
\[
V^{(n)}_+ = \lambda^{2n} V - V^{(n)}_+
\]
then Eq (17) can be written as
\[
V^{(n)}_{x+y} - V^{(n)}_{y} + [U, V^{(n)}_+]_M = V^{(n)}_{x} - V^{(n)}_{y} + [U, V^{(n)}_+)_M.
\]
(20)
It is easy to check that the degrees on the left hand side in (20) \( \geq 0 \), while the degrees on the right hand side in (20) \( \leq 1 \), therefore both sides of (20) are of degree 0 and 1. Thus, we have
\[
V^{(n)}_{x+y} - V^{(n)}_{y} + [U, V^{(n)}_+]_M = (a_{nx} - a_{ny} - 2qcn + 2sana)e_1(0) + (b_{nx} - b_{ny} + 2rcn - 2sbn)e_2(0) + (c_{nx} - c_{ny})e_3(0).
\]
(21)
Let \( V^n = V^{(n)}_+ \), then the zero curvature equation
\[
U_t - U_x + V^n_x - V^n_y + [U, V^n]_M = 0
\]
(22)
or
\[
U_t - U_x = [V^n_x - V^n_y + [U, V^n]_M
\]
(23)
determines the integrable system as following
\[
w_{t_n} - u_x = \begin{pmatrix} \partial_{t_n} - \partial_x & 0 & 0 \\ 0 & \partial_{t_n} - \partial_x & 0 \\ 0 & 0 & \partial_{t_n} - \partial_x \end{pmatrix} \begin{pmatrix} q \\ r \\ s \end{pmatrix}.
\]
(24)
By setting
\[
J_1 = \begin{pmatrix} \partial_{t_n} - \partial_x & 0 & 0 \\ 0 & \partial_{t_n} - \partial_x & 0 \\ 0 & 0 & \partial_{t_n} - \partial_x \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & (\partial_y - \partial_x)^{-1} \end{pmatrix}
\]
and \( G_n = (a_{n+1}, b_{n+1}, c_n)^T \), \( u = (q, r, s)^T \), then (23) can be written as
\[
J_2 J_1 u = G_n.
\]
(25)
If we set
\[
A_{11} = \frac{1}{2}(\partial_y - \partial_x) - s - \frac{1}{2}q(\partial_y - \partial_x)^{-1}r(\partial_y - \partial_x) - q(\partial_y - \partial_x)^{-1}rs
\]
\[
A_{12} = -\frac{1}{2}q(\partial_y - \partial_x)^{-1}q(\partial_y - \partial_x) - q(\partial_y - \partial_x)^{-1}qs
\]
\[
A_{21} = -\frac{1}{2}q(\partial_y - \partial_x)^{-1}q(\partial_y - \partial_x) - r(\partial_y - \partial_x)^{-1}q
\]
\[
A_{22} = \frac{1}{2}(\partial_y - \partial_x) - s - \frac{1}{2}r(\partial_y - \partial_x)^{-1}q(\partial_y - \partial_x) - r(\partial_y - \partial_x)^{-1}qs
\]
\[
A_{31} = -\frac{1}{2}r(\partial_y - \partial_x)^{-1}q(\partial_y - \partial_x)
\]
\[
A_{32} = -\frac{1}{2}r(\partial_y - \partial_x)^{-1}q(\partial_y - \partial_x)
\]
\[
A_{33} = -\frac{1}{2}r(\partial_y - \partial_x)^{-1}q(\partial_y - \partial_x)
\]
and
\[
A = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{pmatrix},
\]
then we have the following result from (18)
\[
G_n = AG_{n-1}.
\]
Hence, (25) can be written as

\[ J_2 J_1 u = G_n = A^n G_0 = A^n \begin{pmatrix} \alpha q \\ \alpha r \\ \alpha s \end{pmatrix} . \quad (27) \]

If we take \( n = 1, \alpha \) = nonzero constant, we obtained (2+1)-dimensional evolution equation from (27)

\[
\begin{align*}
\frac{1}{2} (q_{t_1} - q_x) &= \frac{1}{2} \alpha (\partial_y - \partial_x) q - \alpha q s - \frac{1}{2} \alpha q^2 r, \\
-\frac{1}{2} (t_{t_1} - r_x) &= -\frac{1}{2} \alpha (\partial_y - \partial_x) r - \alpha r s - \frac{1}{2} \alpha r^2 q, \\
(s_{t_1} - s_x) &= -\frac{1}{2} \alpha (\partial_y - \partial_x) (qr).
\end{align*}
\]

(28)

If we take \( n = 2, \alpha \) = nonzero constant, and use identity

\[ q (\partial_y - \partial_x)^2 r - r (\partial_y - \partial_x)^2 q = (\partial_y - \partial_x) [q (\partial_y - \partial_x) r - r (\partial_y - \partial_x) q], \]

we can obtain (2+1)-dimensional evolution equation from (27)

\[
\begin{align*}
\frac{1}{2} (q_{t_2} - q_x) &= \frac{1}{4} (\partial_y - \partial_x)^2 q - \frac{1}{4} (\partial_y - \partial_x) (q^2 s) - \frac{1}{4} (\partial_y - \partial_x) (q^2 r) - \frac{1}{4} s (\partial_y - \partial_x) q \\
&\quad + \frac{1}{4} q [q (\partial_y - \partial_x) r - r (\partial_y - \partial_x) q] + \alpha s^2 q + \frac{3}{2} \alpha sq^2 r + \frac{3}{2} \alpha q^3 r^2 \\
-\frac{1}{2} (t_{t_2} - r_x) &= \frac{1}{4} (\partial_y - \partial_x)^2 r + \frac{1}{4} (\partial_y - \partial_x) (sr) + \frac{1}{2} (\partial_y - \partial_x) (r^2 q) + \frac{1}{2} s (\partial_y - \partial_x) r \\
&\quad + \frac{1}{2} r [q (\partial_y - \partial_x) r - r (\partial_y - \partial_x) q] + \alpha s^2 r + \frac{3}{2} \alpha sr^2 + \frac{3}{2} \alpha q^2 r^3 \\
\frac{1}{2} (s_{t_2} - s_x) &= \frac{1}{4} (\partial_y - \partial_x)^2 q - \frac{1}{4} (\partial_y - \partial_x) (r^2 q) - r (\partial_y - \partial_x) q + \frac{1}{2} r (\partial_y - \partial_x) (sr) \\
&\quad + \frac{1}{2} s (\partial_y - \partial_x) (qr) + \frac{1}{2} q (\partial_y - \partial_x) (sr) + \frac{3}{2} \alpha (\partial_y - \partial_x) (q^2 r^2).
\end{align*}
\]

(29)

If we take \( \partial_y = 0, t_2 = t \), then (29) is reduced to the generalized couple Schrödinger system. We call system (27) a (2+1)-dimensional generalized couple Schrödinger equation.

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