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An Application of Homotopy Perturbation Method for Non-linear Blasius Equation to Boundary Layer Flow Over a Flat Plate

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Abstract: In this article, the problem of an incompressible viscous flow over a flat plate is presented. The homotopy perturbation method (HPM) is employed to solve the well-known Blasius nonlinear differential equation. It has been tried to use a new technique by which one be able to obtain solutions that are very close to the exact solution of the equation. The obtained results have been compared with the exact solution of Blasius equation and another results obtained in previous works so that the high accuracy of results are clear.

Keywords:homotopy perturbation method; incompressible viscous flow; Blasius nonlinear differential equation

1 Introduction

The homotopy perturbation method (HPM) was introduced by He [1-8] in 1998. This method has been used by many mathematicians and engineers to solve various differential equations. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solution. This simple method has been applied to solve linear and non-linear equations of heat transfer [9-11, 22, 23], fluid mechanics [12], non-linear Schrödinger equations [13, 24], some boundary value problems and many other subjects in different disciplines [14-17]. One of the well-known equations arising in fluid mechanics and boundary layer approach is Blasius's differential equation. H. Blasius[18] in 1908 found the exact solution of boundary layer equation over a flat plate. Afterwards it has been solved by Howarth [19] by means of some numerical methods. Since He's homotopy perturbation method (HPM) is a new technique, attempts have been conducted to apply this method for solving Blasius equation [12] and [20]. In this paper, a new initial approximation for the solution is applied by means of Taylor series which has not been used in previous works. Obtained solutions in comparison with previous HPM results provide the higher accuracy.

2 Basic idea of homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following non-linear functional equation:

$$A(U) - f(r) = 0, \qquad r \in \Omega \tag{1}$$

With the following boundary condition:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{2}$$

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where A is a general functional operator, B a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domainΩ. The operator A can be decomposed into two operators L and$N$, where L is linear, and N is nonlinear operator. Eq. (1) can be, therefore, written as follows:

$$L(U) + N(U) - f(r) = 0. \tag{3}$$

Using the homotopy technique, we construct a homotopy $U(r,p) : \Omega \times [0,1] \rightarrow \mathrm{R}$,which satisfies: $H(U,p) = (1-p)[L(U) - L(U_0)] + p[A(U) - f(r)] = 0, \quad p \in [0,1], r \in \Omega$, (4) or

$$H(U,p) = L(U) - L(U_0) + pL(U_0) + p[N(U) - f(r)] = 0, \tag{4}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation for the solution of equation (1), which satisfies the boundary conditions. Obviously, from Eqs.(3) and (4) we will have:

$$H(U,0) = L(U) - L(U_0) = 0, \tag{5}$$

$$H(U,1) = A(U) - f(r) = 0. \tag{6}$$

The changing values of p from zero to unity are just that of $U(r,p)$ from $u_0(r)$ to $u(r)$.In topology, this is called homotopy. According to HPM, we can first use the embedding parameter $p$as a small parameter, and assume that the solution of Eqs. (3) and (4) as a power series in p:

$$U = U_0 + pU_1 + p^2 U_2 + \dots. \tag{7}$$

Setting $p = 1$, results in the approximation to the solution of Eq.(1)

$$U = \lim_{P \to 1} U = U_0 + U_1 + U_2 + \dots. \tag{8}$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated limitations of the traditional perturbation techniques. The series (8) is convergent for more cases. Some criteria are suggested for convergence of the series (8), in [5].

3 Method of solution

Boundary layer flow over a flat plate is governed by the continuity and the Navier-Stokes equations.For a two dimensional, steady state, incompressible flow with zero pressure gradient over a flat plate, governing equations are simplified to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{10}$$

Subjected to boundary conditions:

$$y = 0, \quad u = 0,$$

$$y = \infty, u = U_\infty \quad , \quad \frac{\partial u}{\partial y} = 0. \tag{11}$$

By applying a dimensionless variable (η) defined as:

$$\eta = \frac{y}{\sqrt{x}} Re^{0.5} \quad , \tag{12}$$

(Re is the Reynolds number and defined as: $Re = \frac{U_\infty x}{\nu}$)

The governing equations of (9) and (10) can be reduced to the well-known Blasius equation where f is a function of variable (η):

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0 \tag{13}$$

with boundary equations:

$$\eta = 0, \quad f = \frac{df}{d\eta} = 0,$$

$$\eta \to \infty, \quad \frac{df}{d\eta} = 1. \tag{14}$$

where f is related to u(velocity) by $f' = \frac{u}{U_\infty}$, and the "prime" denotes the derivatives with respect to η. To solve Eq. (13) by homotopy perturbation method, we construct the following homotopy:

$$(1-p)\left(\frac{\partial^3 F}{\partial \eta^3} - \frac{\partial^3 f_0}{\partial \eta^3}\right) + p\left(\frac{\partial^3 F}{\partial \eta^3} + \frac{F}{2}\frac{\partial^2 F}{\partial \eta^2}\right) = 0,$$

or

$$\frac{\partial^3 F}{\partial \eta^3} - \frac{\partial^3 f_0}{\partial \eta^3} + p\left(\frac{\partial^3 f_0}{\partial \eta^3} + \frac{F}{2}\frac{\partial^2 F}{\partial \eta^2}\right) = 0. \tag{15}$$

Suppose that the solution of Eq. (15) to be in the following form:

$$F = F_0 + pF_1 + p^2 F_2 + \dots. \tag{16}$$

Substituting (16) into (15), and some algebraic manipulations and rearranging the coefficients of the terms with identical powers of$p$, we have:

$$\begin{array}{l}
p^0: \ \frac{\partial^3 F_0}{\partial \eta^3} - \frac{\partial^3 f_0}{\partial \eta^3} = 0, \\
p^1: \frac{\partial^3 F_1}{\partial \eta^3} + \frac{\partial^3 f_0}{\partial \eta^3} + \frac{F_0}{2}\frac{\partial^2 F_0}{\partial \eta^2} = 0, \\
p^2: \frac{\partial^3 F_2}{\partial \eta^3} + \frac{F_1}{2}\frac{\partial^2 F_0}{\partial \eta^2} + \frac{F_0}{2}\frac{\partial^2 F_1}{\partial \eta^2} = 0, \\
p^3: \frac{\partial^3 F_3}{\partial \eta^3} + \frac{F_1}{2}\frac{\partial^2 F_1}{\partial \eta^2} + \frac{F_2}{2}\frac{\partial^2 F_0}{\partial \eta^2} + \frac{F_0}{2}\frac{\partial^2 F_2}{\partial \eta^2} = 0, \\
\vdots \ \ \vdots \qquad \vdots
\end{array} \tag{17}$$

First for simplicity we take$F_0 = f_0$. In the present work we start the iteration by defining f_0 as a Taylor series of order two near $\eta = 0$, so that it could be resulted in highly accurate solutions near $\eta = 0$, i.e.

$$F_0 = f_0 = \frac{f''(0)}{2}\eta^2 + f'(0)\eta + f(0). \tag{18}$$

By applying $f''(0)$ =0.332057 from [19] and boundary conditions of Eq. (14) and solving Eq.(17) for F_1, F_2 and F_3 we derive:

$$\begin{aligned}
f_0 &= 0.16602850\eta^2 \\
f_1 &= -0.00045942\eta^5 \\
f_2 &= 0.00000249\ \eta^8 \\
f_3 &= -0.00000001\ \eta^{11}.
\end{aligned} \tag{19}$$

According to (16) and the assumption $p = 1$, we get:

$$f(\eta) = 0.16602850\eta^2 - 0.00045942\eta^5 + 0.00000249\ \eta^8 - 0.00000001\ \eta^{11} \tag{20}$$

Since, Eq. (13) can not be easily solved by the analytical method; it is therefore solved here by homotopy perturbation method using software MAPLE. The results are given in Table 1, and also the results of homotopy perturbation method and numerical method (N.M) are given in [20] and they are presented in this table. Table 2 ,3 are made to compare between present results and results given by Blasius [18]. In Fig. 1, 2 one can also see the comparison between obtained results (Present method) and Blasius's results.

Table 1: Obtained results, in comparison with HPM and numerical methods (N.M) for $f(\eta)$ and $f'(\eta)$

		$f(\eta)$			$f'(\eta)$	
η	Present method	HPM [20]	N.M [20]	Present method	HPM [20]	N.M [20]
0	0	0	0	0	0	0
0.2	0.0066409	0.0069699	0.0066412	0.0664077	0.0696975	0.0664077
0.4	0.0265598	0.0278758	0.0266762	0.1327640	0.1393444	0.1327641
0.6	0.0597345	0.0626959	0.0597215	0.1989370	0.2088105	0.1989372
0.8	0.1061081	0.1113738	0.1061082	0.2647088	0.2778800	0.2647094
1.0	0.1655715	0.1738016	0.1655717	0.3297796	0.3462538	0.3297800
1.2	0.2379484	0.2498038	0.2379487	0.3937756	0.4135539	0.3937761
1.4	0.3229812	0.3391217	0.3229815	0.4562612	0.4793309	0.4562617
1.6	0.4203202	0.4414008	0.4203207	0.5167558	0.5430747	0.5167567
1.8	0.5295172	0.5561797	0.5295180	0.5747553	0.6042289	0.5747581
2.0	0.6500224	0.6828833	0.6500243	0.6297563	0.6622097	0.6297657
2.2	0.7811878	0.8208206	0.7811933	0.6812799	0.7164291	0.6813103
2.4	0.9222734	0.9691873	0.9222901	0.7288906	0.7663226	0.7289819
2.6	1.0724575	1.1270772	1.0725059	0.7722037	0.8113803	0.7724550
2.8	1.2308450	1.2935005	1.2309773	0.8108693	0.8511819	0.8115096
3.0	1.3964712	1.4674133	1.3968082	0.8445213	0.8854328	0.8460444
3.2	1.5682875	1.6477584	1.5690949	0.8726718	0.9140010	0.8760814
3.4	1.7451217	1.8335195	1.7469501	0.8945290	0.9369507	0.9017612
3.6	1.9255896	2.0237911	1.9295251	0.9087097	0.9545718	0.9233296
3.8	2.1079372	2.2178650	2.1160298	0.9128113	0.9673977	0.9411181
4.0	2.2897787	2.4153361	2.3057464	0.9027999	0.9762106	0.9555182

Table 2: Obtained results, in comparison with Blasius's results for $f(\eta)$

	$f(\eta)$	
η	H.P.M (Present method)	Blasius
0	0	0
0.5	0.0415	0.0415
1.0	0.1655	0.1656
1.5	0.3701	0.3701
2.0	0.6500	0.6500
2.5	0.9962	0.9963
3.0	1.3964	1.3968
3.5	1.8350	1.8377
4.0	2.2897	2.3057

Table 3: Obtained results, in comparison with Blasius's results for $f'(\eta)$

	$f'(\eta)$	
η	H.P.M (Present method)	Blasius
0	0	0
0.5	0.1658	0.1659
1.0	0.3298	0.3298
1.5	0.4867	0.4868
2.0	0.6297	0.6298
2.5	0.7511	0.7513
3.0	0.8445	0.8460
3.5	0.9027	0.9130
4.0	0.9028	0.9555

4 Conclusion

In this paper we have studied a well-known Blasius boundary layer equation. We have applied homotopy perturbation method to solve this nonlinear differential equation. Since we have used a Taylor series around$\eta = 0$, we have obtained results with excellent accuracy for $\eta \leq 4$. As the approaches in Tables 1, 2 and Fig .1, the present work result for these values of η benefit more accuracy than [20] and are successfully confirm to the exact solutions of Blasius.

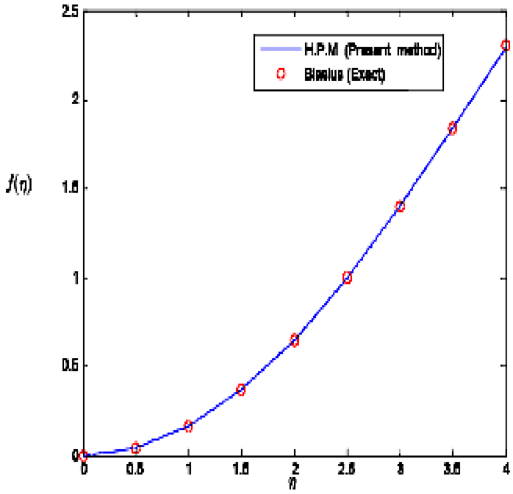

Fig1:The comparison of answers obtained by H.P. M (Present method) and Blasius's results for $f(\eta)$

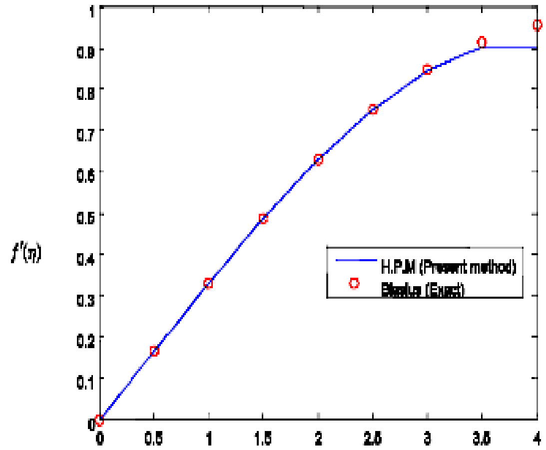

Fig2:The comparison of answers obtained by H.P. M (Present method) and Blasius's results for $f'(\eta)$

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