Homogeneous Balance Method for an Inhomogeneous KdV Equation: Böcklund Transformation and Lax Pair *

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Abstract: Fan proposed an extended homogeneous balance method, and used it to derive auto-Böcklund transformation, Lax pair and explicit solutions of evolutionary equations. In this paper, we applicate it to an inhomogeneous KdV equation.

Keywords: Homogeneous balance method; Lax pair; Böcklund transformation

1 Introduction

In the past years, many powerful and direct methods have been developed to find special solutions of non-linear evolution equations (NEE(s)), such as the inverse method [1], Böcklund transformation [2], Hirota bilinear method [3], the Wronskian determinant technique [4], numerical methods [5] et al. Homogenous balance method (HBM) [6] is proved to be efficient method for finding explicit solutions of NEEs. Fan proposed an extended HBM [8] to get Böcklund transformation.

In the paper, we use Fan’s technique to derive Lax pair and Böcklund transformation for an inhomogeneous KdV equation, and make the HBM more efficient. In the next section, we will express our algorithm.

2 Homogeneous Balance Method

Considering a non-linear evolution equation, with a physical field $u$, and two independent variables $x, t$ as

$$G(u, u_t, u_x, u_{tx}, u_{tt}, u_{xx}, ... ) = 0,$$  \hspace{1cm} (1)

**Step 1.**

Assume

$$u_1 = u_0 + c \frac{\partial^{n+m} \ln(w)}{\partial x^n \partial t^m},$$  \hspace{1cm} (2)

where $w = w(t, x), n \in Z^+, u_0$ is a initial value, and $u_1$ is new solution to Eq.(1).

If we use Fan’s HBM method, we should change non-linear terms into linear ones, while we needn’t do such by means of the simplified form Eq.(2), and make the procedure easier.

Then, we can determine the $c, n$ and $m$ here by balancing the highest degree linear term with the non-linear term.

**Step 2.**

Substituted Eq.(2) into Eq.(1), we derive a series about $w$. Then, making the coefficient of the lowest degree zero, we can gain Böcklund transformation.

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3 Application

The inhomogeneous KdV equation is

$$u_t = 6uu_x + u_{xxx} + (G(t)x + F(t))u_x + 2G(t)u,$$

(3)

where $F(t)$ and $G(t)$ are all analytic functions.

This model with the time-dependent coefficient has been derived to describe a variety of interesting and significant phenomena in non-mEDIATE.

First, balancing $uu_x$ with $u_{xxx}$, $n = 2$ and $m = 0$.

Considering the equation $6uu_x + u_{xxx} = 0$, if $u = e^{2\text{ln}(w)}$, then we can derive $c = 2$.

Second, substitute $u_1 = u_0 + 2\frac{d\text{ln}(w)}{dx}$ into Eq.(3). We omit the equation, and directly extract the coefficients of $w$ with the software Maple.

$$w^{-3} : \quad 4w_x(w_xw_t - 4w_xw_{xxx} + 3w_x^2 - 6w_0w_x^2 - (F(t) + G(t)x)w_x^2) = 0;$$

$$w^{-2} : \quad -2\frac{\partial}{\partial x}(w_{x,x} - 6u_0w_x - F(t)w_x - G(t)xw_x - w_{xxx} - G(t)w_x) - 2w_x(w_xw_t - 4w_xw_x, x + 3w_x^2 - 6u_0w_x^2 - (F(t) + G(t)x)w_x^2) = 0;$$

$$w^{-1} : \quad 2\frac{\partial}{\partial x}(w_{x,t} - 6u_0w_x - F(t)w_x - G(t)xw_x - w_{xxx} - G(t)w_x) = 0;$$

$$w^0 : \quad u_0, t - 6u_0u_0x - u_0w_x = (G(t)x + F(t))u_0x - 2G(t)u_0 = 0.$$

Therefore, the B"{a}cklund transformation of Eq.(3) is

$$u_1 = u_0 + 2\frac{d\text{ln}(w)}{dx},$$

where $u_0$ is a known solution, $w$ satisfies

$$\begin{cases} w_xw_t - 4w_xw_{xxx} + 3w_x^2 - 6u_0w_x^2 - (F(t) + G(t)x)w_x^2 = 0, \\
w_{x,x} - 6u_0w_x - w_{xxx} - (F(t) + G(t)x)w_x = G(t)w_x = 0. \end{cases}$$

(4)

In order to find the Lax pair, we derive the expression of $w_t$ in the first equation, then delete the $w_{t,x}$ in the latter one, and let $w_x = \phi_0^2$, we have $\phi_{x,x} = (\lambda - u)\phi$.

The second equation can be reduced as $\phi_t = (2u_0 + 4\lambda + F(t) + G(t)x)\phi_x + (\frac{G(t)}{2} - u_0, x)\phi$, where $\lambda$ is a analytic function depended on $t$, for detail, $\lambda_t = 2\lambda G(t)$.

Thus, Eq.(3) has the inverse scattering problem

$$\begin{cases} \phi_{x,x} = (\lambda - u_0)\phi, \\
\phi_t = (2u_0 + 4\lambda + F(t) + G(t)x)\phi_x + (\frac{G(t)}{2} - u_0, x)\phi = 0. \end{cases}$$

(5)

4 Summary

There exist three methods to search for Lax pairs of soliton equations: Homogeneous Balance Method, WE’s prolongation theory [8], Franklin Lambert and Johan Springael’s direct method [9]. The method here proves to be efficient and simple in (1+1) dimensional nonlinear evolution equations. With Lax Pairs of soliton equations, many classical nonlinear techniques can be applied, such as Darbroux Transformation [10], Inverse Scattering Transform [11], Hirota method [12].

References


*IJNS homepage:* http://www.nonlinearscience.org.uk/