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# Exact Solutions of the Variable Coefficient Burgers-Fisher Equation with Forced Term

Hongliang Zhang[1] *, Dianchen Lu[2]
[1]School of Mathematical Sciences,Xuzhou Normal University
Xuzhou, Jiangsu, 221116,P.R.China [2]Nonlinear Scientific Research Center, Jiangsu University
Zhenjiang, Jiangsu,212013,P.R.China
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**Abstract:** In this paper, we generalized the Riccati equation method by constructing two new Riccati Equations. With the aid of Mathematica software, we obtained several types of exact solutions for the variable coefficient Burgers-Fisher equation with forced term by using this method,including solitary-wave solutions with constant and variable speeds and periodical solutions.

**Keywords:** Riccati equations; variable coefficient Burgers-Fisher equation; solitary solutions; periodical solutions
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## 1 Introduction

The investigation of the exact solutions to nonlinear evolution equations (NEEs) plays an important role in the study of nonlinear physical phenomena. The constant coefficient nonlinear equation only reflects motion of actual matters change rule approximately, however, the variable coefficient nonlinear equation can describe the property of substance precisely. So it is more interesting to find the solutions for variable coefficient nonlinear equation. A number of methods were presented, such as inverse scattering theory, Hirotas bilinear method, the truncated Painleve expansion, homogeneous balance method, the hyperbolic tangent function series method,the sine-cosine method, and the Jacobi elliptic function expansion method etc[1]-[11].

In this paper, based on the variable coefficient projective Riccati equationmethod [13], we will construct two new Riccati equations to generalize the Riccati equation method, and use this method to solve the variable coefficient Burgers-Fisher equation with forced term, namely,

$$u_t - d(t)u_{xx} + a(t)uu_x + b(t)(u^2 - u) = R(t) \tag{1.1}$$

where $d(t), a(t), b(t)$ and $R(t)$ are arbitrary function of $t$.

It is obvious that Eq.(1.1) is the Burgers-Fisher equation for $R(t) = 0, d(t), a(t)$ and $b(t)$ are constants.When $b(t) = R(t) = 0$,it is variable coefficient Burgers equation, which often emerged in Maths and Physics. For example, for sound wave propagation in medium with viscosity and thermal conductivity, without considering the frequency character and relaxation process of medium on certain condition, the control equation will transform to Burgers equation. When $d(t) = R(t) = 0$, Eq.(1.1) is variable coefficient Fisher equation, which is used to describe nonlinear phenomena such as thermonuclear fusion and hydronium physics etc.

The paper is arranged as follows. In section 2, we briefly describe the improved projective Riccati equation method. In section 3, we obtain several types of solutions of Eq.(1.1) which include solitary-wave solutions and periodical solutions. Finally in section 4, some conclusions are given.

*Corresponding author. *E-mail address*: zhanghongliang-xz@126.com

## 2 Summary of the improved projective Riccati equation method

For the given nonlinear evolution equations(NEEs) with two variables $x$ and $t$

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \cdots) = 0 \tag{2.1}$$

We assume Eq.(2.1) has the following formal solutions:

$$u(x,t) = u(\xi) = \sum_{i=0}^{n} a_i(t) f^i(\xi) + \sum_{j=1}^{n} b_j(t) f^{j-1}(\xi) g^j(\xi) \tag{2.2}$$

where $a_0(t), a_i(t), b_j(t), (i, j = 1, 2, \cdots, n)$ and $\xi = \xi(x,t) = k(t)x + l(t) + \xi_0$ are differential functions with the variable $x$ and $t$. The parameter $n$ can be determined by balancing the highest order derivative terms with the nonlinear terms in Eq.(2.1). $f(\xi), g(\xi)$ satisfy the following projective Riccati equations:

$$(I) f'(\xi) = -q f(\xi) g(\xi), g'(\xi) = q[1 - g^2(\xi) - r f(\xi)], g^2(\xi) = 1 - 2r f(\xi) + (r^2 + \varepsilon) f^2(\xi) \tag{2.3}$$

where "′"denotes $\frac{d}{d\xi}$, $\varepsilon = \pm 1$ , $r, q$ are arbitrary constants. It is easy to see that Eqs.(2.3) admits the following solutions:

$$f_1(\xi) = \frac{a}{b\cosh(q\xi) + c\sinh(q\xi) + ar}, \quad g_1 = \frac{b\sinh(q\xi) + c\cosh(q\xi)}{b\cosh(q\xi) + c\sinh(q\xi) + ar} \tag{2.4}$$

when $\varepsilon = 1$: $a, b, c$ satisfies $c^2 = a^2 + b^2$; when $\varepsilon = -1$: $a, b, c$ satisfies $b^2 = a^2 + c^2$.

$$(II) f'(\xi) = q f(\xi) g(\xi),\ g'(\xi) = q[1 + g^2(\xi) - r f(\xi)],\ g^2(\xi) = -1 + 2r f(\xi) + (1 - r^2) f^2(\xi) \tag{2.5}$$

Eqs.(2.5) has the following solutions:

$$f_2(\xi) = \frac{a}{b\cos(q\xi) + c\sin(q\xi) + ar}, \quad g_2(\xi) = \frac{b\sin(q\xi) - c\cos(q\xi)}{b\cos(q\xi) + c\sin(q\xi) + ar} \tag{2.6}$$

where $a, b, c$ satisfies $a^2 = b^2 + c^2$.

Substituting (2.2) with (2.3) and (2.2) with (2.5) into Eq.(2.1) separately yields a set of differential equations for $f^i(\xi)g^j(\xi)(i, j = 0, 1, 2, \cdots)$. Setting the coefficients of $f^i(\xi)g^j(\xi)$ to zero yields a set of over-determined differential equations(ODEs) in $a_i, b_j, (i = 0, 1, 2, \cdots; j = 1, 2, \cdots)$ and $k, l$. Solving the differential equations(ODEs), we can obtain many exact solutions of Eq.(2.1) according to (2.4) and (2.6).

**Remark 1** Solutions (2.4) and (2.6) contain the results in [12]-[14] completely.

## 3 Exact solutions of the variable coefficient Burgers-Fisher equation with forced term

In order to seek the exact solutions of Eq.(1.1), according to the above step, we define the traveling-wave transformation $\xi = \xi(x,t) = k(t)x + l(t) + \xi_0$,where $k(t)$ and $l(t)$ are arbitrary functions of $t$ and $\xi_0$ is an arbitrary constant. By balancing the highest order derivative terms and the nonlinear terms in Eq.(1.1),we assume Eq.(1.1) has the solution

$$u(x,t) = u(\xi) = a_0(t) + a_1(t) f(\xi) + a_2(t) g(\xi) \tag{3.1}$$

where $\xi(x,t) = k(t)x + l(t) + \xi_0, \xi_0$ are arbitrary constants.

State 1:

Substituting(2.3) with (3.1) into (1.1) and setting the coefficients of $f^i(\xi)g^j(\xi)(i = 0, 1, 2, \cdots; j = 0, 1)$ to zero yields an ODEs. Solving this ODEs,we have

Case 1:

$$d(t) = C_0 b(t), k(t) = k_0, a_0(t) = \int R(t)dt + C_1, a_1(t) = \pm 3C_0 k_0^2 q^2, a_2(t) = 0,$$
$$l(t) = -\int a(t) k_0 (\int R(t)dt + C_1)dt + C_2, r = \pm 1, \varepsilon = -1,$$

where $R(t), d(t)$ and $a(t)$ satisfies the restrict conditions

$$\int R(t)dt + C_1 = \frac{1}{2} - \frac{1}{2}C_0k_0^2q^2, 3d(t)k_0q^2r + 3C_0d(t)k_0^3q^4r - a^2(t) = 0$$

and $k_0 \neq 0, q \neq 0, C_0 \neq 0, C_1, C_2$ are constants.
Case 2:

$$\begin{aligned} &d(t) = C_3a(t), k(t) = k_0, a_0(t) = \int R(t)dt + C_1, a_1(t) = 0, a_2(t) = C_3qk_0, \\ &l(t) = \int (C_3k_0b(t) + \tfrac{k_0a(t)}{2})dt + C_4 \end{aligned}$$

where $R(t)$ and $b(t)$ satisfies the restrict conditions

$$\int R(t)dt + C_1 = -\frac{1}{2} + \sqrt{-3C_3^2b(t)k_0^2q^2 + \frac{1}{4}}$$

and $k_0 \neq 0, q \neq 0, C_0 \neq 0, C_3, C_3, r$ are constants.
Case 3:

$$\begin{aligned} &d(t) = C_0b(t), k(t) = k_0, a_0(t) = \int R(t)dt + C_1, a_1(t) = \pm 3C_0k_0^2q^2, a_2(t) = 0, \\ &l(t) = -\int a(t)k_0(\int R(t)dt + C_1)dt + C_2, r = \pm i, \varepsilon = -1, \end{aligned}$$

where $R(t), d(t)$ and $a(t)$ satisfies the restrict conditions

$$\int R(t)dt + C_1 = \frac{1}{2} - \frac{1}{2}C_0k_0^2q^2, 3d(t)k_0q^2r + 3C_0d(t)k_0^3q^4r - a^2(t) = 0$$

and $k_0 \neq 0, q \neq 0, C_0 \neq 0, C_1, C_2$ are constants.

Substituting above solutions into (3.1), using (2.4), we obtain the following like-solitary solutions of Eq.(1.1):

$$u_{11}(x,t) = u_{11}(\xi_{11}) = \int R(t)dt + C_1 \pm \frac{3aC_0k_0^2q^2}{b\cosh(q\xi_{11}) + c\sinh(q\xi_{11}) \pm a}$$

$$u_{12}(x,t) = u_{12}(\xi_{12}) = \int R(t)dt + C_1 + C_3qk_0\frac{b\sinh(q\xi_{12}) + c\cosh(q\xi_{12})}{b\cosh(q\xi_{12}) + c\sinh(q\xi_{12}) + ar}$$

$$u_{13}(x,t) = u_{13}(\xi_{13}) = \int R(t)dt + C_1 \pm \frac{3aC_0^2k_0^2q^2}{b\cosh(q\xi_{13}) + c\sinh(q\xi_{13}) \pm ai}$$

where

$$\begin{aligned} &\xi_{11} = \xi_{13} = k_0x - \int a(t)k_0(\int R(t)dt + C_1)dt + C_2, \\ &\xi_{12} = k_0x + \int (C_3k_0b(t) + \tfrac{k_0a(t)}{2})dt + C_4 \end{aligned}$$

State 2:

In the same manner,substituting (2.5) with (3.1) into (1.1) and setting the coefficients of $f^i(\xi)g^j(\xi)(i = 0, 1, 2, \cdots;$ $j = 0, 1)$ to zero yields an ODEs. Solving this ODEs,we have
Case 4:

$$\begin{aligned} &d(t) = C_0b(t), k(t) = k_0, a_0(t) = \int R(t)dt + C_1, a_1(t) = \pm 3C_0k_0^2q^2r, a_2(t) = 0, \\ &l(t) = -\int a(t)k_0(\int R(t)dt + C_1)dt + C_2, r = \pm 1, \end{aligned}$$

where $R(t), d(t)$ and $a(t)$ satisfies the restrict conditions

$$\int R(t)dt + C_1 = \frac{1}{2} + \frac{1}{2}C_0k_0^2q^2, -3d(t)k_0q^2rb(t) + 3d^2(t)k_0^3q^2ra^2(t) - a^2(t) = 0$$

and $k_0 \neq 0, q \neq 0, C_0 \neq 0, C_1, C_2$ are constants.
Case 5:

$$\begin{aligned} &d(t) = C_3a(t), k(t) = k_0, a_0(t) = \int R(t)dt + C_1, a_1(t) = 0, a_2(t) = -C_3qk_0, \\ &l(t) = \int (C_3k_0b(t) - \tfrac{k_0a(t)}{2})dt + C_4 \end{aligned}$$

where $R(t)$ satisfiy the restrict conditions

$$\int R(t)dt + C_1 = \frac{1}{2} \pm \sqrt{-C_3^2k_0^2q^2 + \frac{1}{4}}$$

and $k_0 \neq 0, q \neq 0, C_3 \neq 0, C_1, C_4, r$ are constants.
Case 6:

$$\begin{array}{l} d(t) = C_3 a(t), k(t) = k_0, a_0(t) = \int R(t)dt + C_1, a_1(t) = 0, a_2(t) = -2C_3 k_0 q, \\ l(t) = \int (2C_3 k_0 b(t) - \frac{k_0 a(t)}{2})dt + C_5, r = 0 \end{array}$$

where $R(t)$ satisfiy the restrict conditions

$$\int R(t)dt + C_1 = \frac{1 \pm \sqrt{1 + 16C_3^2 k_0^2 q^2}}{2}$$

and $k_0 \neq 0, q \neq 0, C_3 \neq 0, C_1, C_5$ are constants.

Substituting above solutions into (3.1), using (2.6), we obtain the following like-periodic solutions of Eq.(1.1):

$$u_{21}(x,t) = u_{21}(\xi_{21}) = \int R(t)dt + C_1 \mp \frac{3aC_0 k_0^2 q^2}{b\cos(q\xi_{21}) + c\sin(q\xi_{21}) \pm a}$$

$$u_{22}(x,t) = u_{22}(\xi_{22}) = \int R(t)dt + C_1 - \frac{C_3 q k_0 [b\sin(q\xi_{22}) - c\cos(q\xi_{22})]}{b\cos(q\xi_{22}) + c\sin(q\xi_{22}) + ar}$$

$$u_{23}(x,t) = u_{23}(\xi_{23}) = \int R(t)dt + C_1 - \frac{2C_3 q k_0 [b\sin(q\xi_{23}) - c\cos(q\xi_{23})]}{b\cos(q\xi_{23}) + c\sin(q\xi_{23})}$$

where

$$\begin{array}{l} \xi_{21} = k_0 x - \int a(t) k_0 (\int R(t)dt + C_1)dt + C_2 \\ \xi_{22} = k_0 x + \int (C_3 k_0 b(t) - \frac{k_0 a(t)}{2})dt + C_4 \\ \xi_{23} = k_0 x + \int (2C_3 k_0 b(t) - \frac{k_0 a(t)}{2})dt + C_5 \end{array}$$

**Remark 2** When $R(t) = 0$, if we choose the different values of $a; b; c; r$, we can obtain more solitary wave solutions with variable speed and singular periodical solutions with variable speed.The solution $u_{12}$ contain the results in [11] and [15] completely.

# 4 Conclusion

In this paper, we constructed two new Riccati equations and successfully applied it to variable coefficient Burgers-Fisher with forced term. We also obtained many different solutions, including solitary-wave solutions with variable speed and periodical solutions. To our knowledge, so far these solutions are not reported yet. How our method is applied to treat more complicated other kinds of nonlinear PDEs is now under investigation.

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