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Hausdorff Dimension of Fractals Associated to Series

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Abstract: In this paper we describe a natural way to associate fractals to a certain class of absolutely convergent series in $\mathbb{R}^n$. We give a sufficient condition to calculate the Hausdorff dimension of some fractals with overlapping.

Keywords: fractal; overlapping; Hausdorff dimension

1 Introduction

This paper describes a natural way to associate fractals to a certain class of absolutely convergent series in $\mathbb{R}^n$. Under suitable conditions on the series, we give the Hausdorff dimension of these fractals. This result can be applied to some standard fractals such as λ-Cantor set ([11]), Sierpinski gasket and McMullen set, etc.

Let $\Omega = \{0, 1, 2, \ldots, N-1\}$, $D = \{D_0, D_1, \ldots, D_{N-1}\} \subset M_n(\mathbb{R})$ is a set of matrices where $D_0 = 0$. For $A \in M_n(\mathbb{R})$, let $\|A\|$ denote the norm of A, i.e $\|A\| = \sup_{|x|=1, x\in\mathbb{R}^n} |Ax|$ where $|x|$ is the Euclidean norm of x. Let $\sum_{k=1}^{\infty} a_k$ be an absolutely convergent series of vectors in $\mathbb{R}^n$, i.e., $\sum_{k=1}^{\infty} |a_k| < \infty$. We define a mapping $\pi : \Omega^\infty \to \mathbb{R}^n$ by putting

$$\pi(\sigma) = \sum_{k=1}^{\infty} D_{\sigma_k} a_k, \ \ \forall\, \sigma \in \Omega^\infty,$$

where $\Omega^\infty = \{0, 1, 2, \ldots, N-1\}^\infty$ is a symbol space. For every positive integer p, $\Omega^p = \{(\omega_1 \cdots \omega_p) : \omega_i \in \Omega\}$ is the set of all finite words with length p. It is interesting in the properties of the image $K = \pi(\Omega^\infty)$, such as Box-counting measure and Box dimension of K. For more details of Box-counting measure and Box dimension, we refer to [1, 2]. We may also consider Lipschitz equivalence between fractals generated by two absolutely convergent series, for more details on Lipschitz equivalence, we refer to [3, 4]. In present paper, we only consider Hausdorff dimension of K. It is easy to show that the mapping π is continuous, hence K is a compact subset of $\mathbb{R}^n$. If we take $D = \{0, I_n\}$ where I_n denotes $n \times n$ identity matrix, K can be generated by the following method. Let $\mathcal{P}(\mathbb{N})$ be the set of all subsets of $\mathbb{N}$. We define a mapping $\phi : \mathcal{P}(\mathbb{N}) \to \mathbb{R}^n$ by

$$\phi(\emptyset) = 0 \ \text{ and } \ \phi(A) = \sum_{k\in A} a_k \ \text{ for all } \ A \neq \emptyset,$$

then $\pi(\Omega^\infty) = \phi(\mathcal{P}(\mathbb{N}))$ in this case. For more details on this method, we refer to [6, 7].

In this paper, we give a sufficient condition to calculate the Hausdorff dimension of the fractals defined above, especially, the result can be used to deal with the self-similar sets with overlapping such as the λ-Cantor set. In Section 3, we give some examples of classical fractals. Throughout this paper, $\dim_H K$ denotes the Hausdorff dimension of the set K; for definition of Hausdorff dimension, and for the definitions of Hausdorff measures $\mathcal{H}^\tau_\delta(K)$, $\mathcal{H}^s(K)$, see [5, 8–10].

Let $R_p = \sum_{k\geq p+1} |a_k|$ and

$$d_p = \frac{1}{2} \min \left\{ |\pi(\sigma * 0^\infty) - \pi(\tau * 0^\infty)| : \sigma, \tau \in \Omega^p, \sigma \neq \tau \right\}.$$

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Theorem 1.1 *If $d_p > 0$ for all integer p and $\lim_{p\to\infty} \frac{\log(d_p+R_p)}{\log d_p} = 1$, then*

$$\dim_H K = \liminf_{k\to\infty} \frac{k\log N}{-\log R_k}.$$

2 Proof of Theorem 1.1

Write $M_D = \max\{\|A-B\| : A, B \in D\}$, $K_\omega = \pi([\omega])$ for $\omega \in \Omega^* = \cup_{p=0}^{\infty}\Omega^p$ where $[\omega] \subset \Omega^\infty$ is the cylinder with respect to ω. Keep the assumption of Theorem 1.1.

Lemma 2.1 *For each $x \in \mathbb{R}^n$, $p \in \mathbb{N}$, we have*

$$\#\{\omega : \omega \in \Omega^p \text{ such that } B(x,r) \cap K_\omega \neq \emptyset\} \le C_{p,r},$$

where $C_{p,r} = \left(\frac{r+d_p+M_D R_p}{d_p}\right)^n$.

Proof. Write $\widetilde{\Omega}^p_{x,r} := \{\omega : \omega \in \Omega^p$ such that $B(x,r)\cap K_\omega \neq \emptyset\}$. Notice that $|K_\omega| \le M_D R_p$ for $\omega \in \Omega^p$ where $|K_\omega|$ denotes the diameter of K_ω, hence the balls $B(\pi(\omega * 0^\infty), d_p)$ with $\omega \in \widetilde{\Omega}^p$, are contained in the ball $B(x, r+M_D R_p + d_p)$. By the definition of d_p, these balls have interiors pairwise disjoint, thus the sum of volumes of these balls does not exceed the volume of $B(x, r+M_D R_p + d_p)$. Hence we have

$$\#\widetilde{\Omega}^p_{x,r} \cdot (d_p)^n \le (r + d_p + M_D R_p)^n,$$

which yields the result. ■

Corollary 2.1 *If $r \le R_P \le d_p$, then $C_{p,r} \le (2+M_D)^n$.*

Proof. The proof is straightforward by Lemma 2.1 ■

Corollary 2.2 *For any $\varepsilon > 0$, there exists an integer $p_0 \in \mathbb{N}$ such that $C_{p,r} \le (1+M_D)^n d_p^{-\varepsilon}$ whenever $p \ge p_0,\ r \le R_p$ and $d_p \le R_p$.*

Proof. Since $\lim_{p\to\infty} \frac{\log(d_p+R_p)}{\log d_p} = 1$, for any $\varepsilon \ge 0$, there exists an integer $p_0 \in \mathbb{N}$ such that for any

$$p \ge p_0,\ d_p + R_p \le d_p^{1-\frac{\varepsilon}{n}}.$$

If $r \le R_p$ and $d_p \le R_p$, then

$$C_{p,r} \le \left(\frac{(1+M_D)(R_p+d_p)}{d_p}\right)^n \le (1+M_D)^n d_p^{-\varepsilon}.$$

■

Proof of Theorem 1.1

(1)Write $\liminf_{k\to\infty} \frac{k\log N}{-\log R_k} = s$, take $t > s$ and fix $0 < \delta < M_D$. Take p such that $M_D R_p \le \delta$ and $t > \frac{p\log N}{-\log R_p}$, then $\{K_\omega : \omega \in \Omega^p\}$ is a δ-covering. For this covering, We have

$$\sum_{\omega\in\Omega^p} |K_\omega|^t \le N^p (M_D R_p)^t < M_D^t.$$

Therefore

$$\mathcal{H}^t_\delta(K) < M_D^t,$$

letting $\delta \to 0$, we get

$$\mathcal{H}^t(K) \le M_D^t.$$

As this holds for all $t > s$,

$$\dim_H(K) \le s. \tag{1}$$

(2) If $s=0$, by (1), $\dim_H(K)=s=0$. If $s>0$, we define a mass distribution μ on Ω^∞ as follows

$$\mu([\omega]) = 2^{-|\omega|},\ \forall\, \omega \in \Omega^*.$$

Therefore we can define a mass distribution ν on K

$$\nu(B) = \mu(\pi^{-1}(B)) \text{ for all Borel set } B \subset \mathbb{R}^n.$$

For any $0<\varepsilon<\frac{s}{3}$, there exists $p_1 \in \mathbb{N}$ such that $N^{-p} \le R_p^{s-\varepsilon}$ for all $p \ge p_1$.

Take any ball $B(x,r)$ with small r and let p be the number such that $R_{p+1} < r \le R_p$ and $p > \max(p_0, p_1)$.

(i) If $d_p \le R_p$, it follows from Corollary 2.2 that $C_{p,r} \le (1+M_D)^n d_p^{-\varepsilon}$. Since

$$\lim_{p\to\infty} \frac{\log(d_p+R_p)}{\log d_p} = 1,$$

which implies $\frac{\log(R_p)^{-\varepsilon}}{\log d_p^{-\varepsilon}} > \frac{1}{2}$ for large p, that is, $d_p^{-\varepsilon} < (R_p)^{-2\varepsilon}$. So we have

$$\begin{aligned} & \nu(B(x,r)) = \nu(B(x,r)\cap K) = \mu\left(\cup_{\omega\in\widetilde{\Omega}^p}[\omega]\right) \\ \le\ & \#\widetilde{\Omega}^p_{x,r} \cdot N^{-p} \le (1+M_D)^n \cdot (d_p^{-\varepsilon}) \cdot (N N^{-(p+1)}) \\ \le\ & (1+M_D)^n \cdot (R_p^{-2\varepsilon}) \cdot (N R_{p+1}^{s-\varepsilon}) \\ \le\ & (1+M_D)^n N R_{p+1}^{s-3\varepsilon} \\ \le\ & N(1+M_D)^n r^{s-3\varepsilon}. \end{aligned}$$

(ii) If $d_p \ge R_p$, by Corollary 2.1, we have $C_{p,r} \le (2+M_D)^n$. Thus

$$\begin{aligned} & \nu(B(x,r)) = \nu(B(x,r)\cap K) = \mu\left(\cup_{\omega\in\widetilde{\Omega}^p_{x,r}}[\omega]\right) \le \#\widetilde{\Omega}^p_{x,r}\cdot N^{-p} \\ \le\ & (2+M_D)^n \cdot N^{-p} \le (2+M_D)^n N R_{p+1}^{s-\varepsilon} \le (2+M_D)^n N r^{s-\varepsilon}. \end{aligned}$$

Take $C=(2+M_D)^n N$, then

$$\nu(B(x,r)) \le C r^{s-3\varepsilon}.$$

By the principle of mass distribution, we have $\dim_H K \ge s-3\varepsilon$, which yields $\dim_H K \ge s$.

By (1), $\dim_H K = s$, the proof is completed.

Corollary 2.3 *If* $\sup \frac{R_p}{d_p} < +\infty$*, then* $\dim_H K = \liminf_{k\to\infty} \frac{k\log N}{-\log R_k}$.

Proof. It is clear that $\sup \frac{R_p}{d_p} < +\infty$ implies $\lim_{p\to\infty} \frac{\log(d_p+R_p)}{\log d_p} = 1$ which completes the proof. ■

Corollary 2.4 *Let* $D=\{0, I_n\}$ *where* I_n *is* $n\times n$ *identity matrix. If* $|a_p| \ge R_p$ *for large* p*, then* $\dim_H K = \liminf_{k\to\infty} \frac{k\log 2}{-\log R_k}$

Proof. For any distinct $\sigma,\ \tau \in \Omega^p$, we have

$$|\pi(\sigma * 0^\infty) - \pi(\sigma * 0^\infty)| = |\sum_{i=1}^p \mathcal{E}_i a_i|$$

where $\mathcal{E}_i \in \{0, -I_n, I_n\}$. Let $k=\min\{i : \sigma_i \ne \tau_i,\ 1 \le i \le p\}$, thus

$$\begin{aligned} & |\pi(\sigma * 0^\infty) - \pi(\sigma * 0^\infty)| \\ \ge\ & |a_k| - |\sum_{i=k+1}^p \mathcal{E}_i a_i| \\ \ge\ & |a_k| - \sum_{i=k+1}^p |a_i| \\ =\ & |a_k| - R_k + R_p. \end{aligned}$$

Therefore, $2d_p \ge R_p$.

By Corollary 2.3, $\dim_H K = \liminf_{k\to\infty} \frac{k\log 2}{-\log R_k}$. ■

Remark 2.1 *Corollary 2.4 implies Theorem 3 in [6] where the condition is* $\sup \frac{R_p}{|a_p|} < 1$.

3 Some examples

Example 3.1 *Take the series* $\sum_{k=1}^{\infty} \frac{2}{3^k}$, $D = \{0, 1\}$, *then the corresponding fractal is the classical middle-third Cantor set. Since* $a_k = \frac{2}{3^k} > \frac{1}{3^k} = R_k$, *by Corollary 2.4, we have*

$$\dim_H K = \liminf_{k\to 0} \frac{k \log 2}{-\log R_k} = \frac{\log 2}{\log 3}.$$

Example 3.2 *Take the series* $\sum_{k=1}^{\infty} \frac{1}{3^k}$, $D = \{0, \lambda, 2\}$, $\lambda \in [0, 1]$, *then the corresponding fractal is the* λ*-Cantor set. Let* $\lambda = \frac{q}{m} \in \mathbb{Q}_{nc}$ *as in [11], then* $m(D - D) \subset \mathbb{Z}$ *and* $d_p \geq \frac{3^{-p}}{2m} > 0$ *for all* $p \in \mathbb{N}$. *Since* $R_p \leq \frac{1}{3^p}$, *it follows that* $\frac{R_p}{d_p} \leq 2m$. *By Corollary 2.3, we have*

$$\dim_H K = \liminf_{k\to 0} \frac{k \log 3}{-\log R_k} = \frac{\log 3}{\log 3} = 1.$$

Example 3.3 *Take the series* $\sum_{k=1}^{\infty} a_k$ *with* $a_k = \begin{pmatrix} \frac{1}{2^k} \\ 0 \end{pmatrix} \in \mathbb{R}^2$ *and*

$$D = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \right\},$$

then the corresponding fractal is the Sierpinski triangle or gasket. It is easy to check that $d_p \geq \frac{1}{2^{p+1}}$ *and* $R_p = \frac{1}{2^p}$, *which yields* $\frac{R_p}{d_p} \leq 2$. *By Corollary 2.3,*

$$\dim_H K = \liminf_{k\to 0} \frac{k \log 3}{-\log R_k} = \frac{\log 3}{\log 2}.$$

Example 3.4 *Take the series* $\sum_{k=1}^{\infty} a_k$ *with* $a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *and* $a_k = \begin{pmatrix} \frac{1}{4^k} \\ \frac{1}{4^k} \prod_{i=2}^{k} (1 - \frac{1}{i}) \end{pmatrix}$ *for* $k \geq 2$,

$$D = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\}.$$

It is easy to check that $R_p = \frac{1}{3} \left(\frac{1}{4^p}\right)$ *and*

$$c_1 \left(\frac{1}{4^p} \prod_{i=2}^{p} \left(1 - \frac{1}{i}\right) \right) \leq d_p \leq c_2 \left(\frac{1}{4^p} \prod_{i=2}^{p} \left(1 - \frac{1}{i}\right) \right)$$

where c_1, c_2 *are positive constants. Thus* $\lim_{p\to 0} \frac{\log R_p}{\log d_p} = 1$ *which yields* $\lim_{p\to 0} \frac{\log(d_p + R_p)}{\log d_p} = 1$. *By Theorem 1.1, the Hausdorff dimension of the corresponding fractal* K *is*

$$\dim_H K = \liminf_{p\to 0} \frac{p \log 4}{-\log R_p} = \frac{\log 4}{\log 4} = 1.$$

However, $\frac{R_p}{d_p} \to \infty$ *as* $p \to \infty$ *which illustrate that the condition of Corollary 2.3 is stronger than that of Theorem 1.1.*

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References

[1] G. X. Dai, Y. Liu and Z. G. Feng. On box dimension of profile curves of SPS. *International Journal of Nonlinear Science*, 2(3):(2006),188-192.

[2] T. Peng and Z. G. Feng. The box-counting measure of the star product surface. *International Journal of Nonlinear Science*, 6(3):(2008),281-288.

[3] M. F. Dai and X. Liu. Lipschitz equivalence between two Sierpinski gasketse. *International Journal of Nonlinear Science*, 2(2):(2006),77-82.

[4] Y. Xiong and L. f. Xi. A counterexample on gap property of bi-Lipschitz constants. *International Journal of Nonlinear Science*, 7(23):(2009),368-370.

[5] P. Mattila. Lecture Notes on Geometry Measure Theory.*Universidad de Extremadura* (1986).

[6] M. Moran. Fractal series. *Mathematika*, 36:(1989),334-348.

[7] M. Moran. Dimension function for fractal sets associated to series. *Proceedings of the American Mathematical Society*, 120:(1994),749-754.

[8] K. J. Falconer. Fractal Geometry. Mathematical Foundations and Applications.*John Wiley & Sons* (1990).

[9] K. J. Falconer. Techniques in Fractal Geometry.*John Wiley and Sons, Ltd., Chichester*(1997).

[10] C. A. Rogers. Hausdorff Measures. *Cambridge University Press*(1970).

[11] Hui Rao and Zhiying Wen. Some studies of a class of self-similar fractals with overlap structure. *Adv. appl. Math.*, 20:(1998),50-72.