Hausdorff Dimension of Fractals Associated to Series

Daoxin Ding\(^1\), Qin Wang\(^2\), Lifeng Xi\(^3\) *

\(^1\) School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, P. R. China
\(^2\) Department of Computer Science, Zhejiang Wanli University, Ningbo, 315100, P. R. China
\(^3\) Institute of Mathematics, Zhejiang Wanli University, Ningbo 315100, P. R. China

(Received 10 December 2009, accepted 22 January 2010)

Abstract: In this paper we describe a natural way to associate fractals to a certain class of absolutely convergent series in \(\mathbb{R}^n\). We give a sufficient condition to calculate the Hausdorff dimension of some fractals with overlapping.

Keywords: fractal; overlapping; Hausdorff dimension

1 Introduction

This paper describes a natural way to associate fractals to a certain class of absolutely convergent series in \(\mathbb{R}^n\). Under suitable conditions on the series, we give the Hausdorff dimension of these fractals. This result can be applied to some standard fractals such as \(\lambda\)-Cantor set ([11]), Sierpinski gasket and McMullen set, etc.

Let \(\Omega = \{0, 1, 2, \ldots, N - 1\}\), \(D = \{D_0, D_1, \ldots, D_{N-1}\} \subset M_n(\mathbb{R})\) is a set of matrices where \(D_0 = 0\). For \(A \in M_n(\mathbb{R})\), let \(|A|\) denote the norm of \(A\), i.e., \(|A| = \sup_{x, x \in \mathbb{R}^n} |Ax|\) where \(|x|\) is the Euclidean norm of \(x\). Let \(\sum_1^\infty a_k\) be an absolutely convergent series of vectors in \(\mathbb{R}^n\), i.e., \(\sum_1^\infty |a_k| < \infty\). We define a mapping \(\pi: \Omega^\infty \rightarrow \mathbb{R}^n\) by putting

\[
\pi(\sigma) = \sum_{k=1}^\infty D_{\sigma_k} a_k, \ \forall \sigma \in \Omega^\infty,
\]

where \(\Omega^\infty = \{0, 1, 2, \ldots, N - 1\}^\infty\) is a symbol space. For every positive integer \(p\), \(\Omega^p = \{\omega_1 \cdots \omega_p : \omega_i \in \Omega\}\) is the set of all finite words with length \(p\). It is interesting in the properties of the image \(K = \pi(\Omega^\infty)\), such as Box-counting measure and Box dimension of \(K\). For more details of Box-counting measure and Box dimension, we refer to [1, 2]. We may also consider Lipschitz equivalence between fractals generated by two absolutely convergent series, for more details on Lipschitz equivalence, we refer to [3, 4]. In present paper, we only consider Hausdorff dimension of \(K\). It is easy to show that the mapping \(\pi\) is continuous, hence \(K\) is a compact subset of \(\mathbb{R}^n\). If we take \(D = \{0, I_n\}\) where \(I_n\) denotes \(n \times n\) identity matrix, \(K\) can be generated by the following method. Let \(P(\mathbb{N})\) be the set of all subsets of \(\mathbb{N}\). We define a mapping \(\phi: P(\mathbb{N}) \rightarrow \mathbb{R}^n\) by

\[
\phi(\emptyset) = 0 \text{ and } \phi(A) = \sum_{k \in A} a_k \text{ for all } A \neq \emptyset,
\]

then \(\pi(\Omega^\infty) = \phi(P(\mathbb{N}))\) in this case. For more details on this method, we refer to [6, 7].

In this paper, we give a sufficient condition to calculate the Hausdorff dimension of the fractals defined above, especially, the result can be used to deal with the self-similar sets with overlapping such as the \(\lambda\)-Cantor set. In Section 3, we give some examples of classical fractals. Throughout this paper, \(\dim_H K\) denotes the Hausdorff dimension of the set \(K\); for definition of Hausdorff dimension, and for the definitions of Hausdorff measures \(H^p_{\delta}(K)\), \(H^p(K)\), see [5, 8–10].

Let \(R_p = \sum_{k \geq p+1} |a_k|\) and

\[
d_p = \frac{1}{2} \min \{|\pi(\sigma * 0^\infty) - \pi(\tau * 0^\infty)| : \sigma, \tau \in \Omega^p, \sigma \neq \tau\}.
\]

* Corresponding author. E-mail address: xllf@zwu.edu.cn
Theorem 1.1 If $d_p > 0$ for all integer $p$ and $\lim_{p \to \infty} \frac{\log(d_p + R_p)}{\log d_p} = 1$, then

$$\dim_H K = \liminf_{k \to \infty} \frac{k \log N}{-\log R_k}.$$ 

2 Proof of Theorem 1.1

Write $M_D = \max\{\|A - B\| : A, B \in D\}$, $K_\omega = \pi([\omega])$ for $\omega \in \Omega^* = \cup_{p=0}^\infty \Omega^p$ where $[\omega] \subset \Omega^\infty$ is the cylinder with respect to $\omega$. Keep the assumption of Theorem 1.1.

Lemma 2.1 For each $x \in \mathbb{R}^n$, $p \in \mathbb{N}$, we have

$$\#\{\omega : \omega \in \Omega^p \text{ such that } B(x, r) \cap K_\omega \neq \emptyset\} \leq C_{p,r},$$

where $C_{p,r} = \left(\frac{r + d_p + M_D R_p}{d_p}\right)^n$.

Proof. Write $\tilde{\Omega}_{x,r}^D := \{\omega : \omega \in \Omega^p \text{ such that } B(x, r) \cap K_\omega \neq \emptyset\}$. Notice that $|K_\omega| \leq M_D R_p$ for $\omega \in \Omega^p$ where $|K_\omega|$ denotes the diameter of $K_\omega$, hence the balls $B(\pi(\omega + 0^\infty), d_p)$ with $\omega \in \tilde{\Omega}_{x,r}^D$, are contained in the ball $B(x, r + M_D R_p + d_p)$. By the definition of $d_p$, these balls have interiors pairwise disjoint, thus the sum of volumes of these balls does not exceed the volume of $B(x, r + M_D R_p + d_p)$. Hence we have

$$\#\tilde{\Omega}_{x,r}^D \cdot (d_p)^n \leq (r + d_p + M_D R_p)^n,$$

which yields the result.

Corollary 2.1 If $r \leq R_P \leq d_p$, then $C_{p,r} \leq (2 + M_D)^n$.

Proof. The proof is straightforward by Lemma 2.1

Corollary 2.2 For any $\varepsilon > 0$, there exists an integer $p_0 \in \mathbb{N}$ such that $C_{p,r} \leq (1 + M_D)^n d_p^{-\varepsilon}$ whenever $p \geq p_0$, $r \leq R_p$ and $d_p \leq R_p$.

Proof. Since $\lim_{p \to \infty} \frac{\log(d_p + R_p)}{\log d_p} = 1$, for any $\varepsilon \geq 0$, there exists an integer $p_0 \in \mathbb{N}$ such that for any

$$p \geq p_0, \quad d_p + R_p \leq d_p^{1 - \varepsilon}.$$

If $r \leq R_p$ and $d_p \leq R_p$, then

$$C_{p,r} \leq \left(\frac{(1 + M_D)(R_p + d_p)}{d_p}\right)^n \leq (1 + M_D)^n d_p^{-\varepsilon}.$$

Proof of Theorem 1.1

(1) Write $\liminf_{k \to \infty} \frac{k \log N}{-\log R_k} = s$, take $t > s$ and fix $0 < \delta < M_D$. Take $p$ such that $M_D R_p \leq \delta$ and $t > \frac{p \log N}{-\log R_p}$, then $\{K_\omega : \omega \in \Omega^p\}$ is a $\delta$-covering. For this covering, We have

$$\sum_{\omega \in \Omega^p} |K_\omega|^t \leq N^t (M_D R_p)^t < M_D^t.$$

Therefore

$$\mathcal{H}^t_s(K) < M_D^t,$$

letting $\delta \to 0$, we get

$$\mathcal{H}^t(K) \leq M_D^t.$$

As this holds for all $t > s$,

$$\dim_H(K) \leq s.$$ 

IUNS email for contribution: editor@nonlinearscience.org.uk
(2) If \( s = 0 \), by (1), \( \dim_H(K) = s = 0 \). If \( s > 0 \), we define a mass distribution \( \mu \) on \( \Omega^\infty \) as follows

\[
\mu([\omega]) = 2^{-|\omega|}, \forall \omega \in \Omega^*.
\]

Therefore we can define a mass distribution \( \nu \) on \( K \)

\[
\nu(B) = \mu(\pi^{-1}(B)) \text{ for all Borel set } B \subset \mathbb{R}^n.
\]

For any \( 0 < \varepsilon < \frac{1}{2} \), there exists \( p_1 \in \mathbb{N} \) such that \( N^{-p} \leq R_{p}^{-\varepsilon} \) for all \( p \geq p_1 \).

Take any ball \( B(x, r) \) with small \( r \) and let \( p \) be the number such that \( R_{p+1} < r \leq R_p \) and \( p > \max(p_0, p_1) \).

(i) If \( d_p \leq R_p \), it follows from Corollary 2.2 that \( C_{p, r} \leq (1 + M_D)^n d_p^{-\varepsilon} \). Since

\[
\lim_{p \to \infty} \frac{\log(d_p + R_p)}{\log d_p} = 1,
\]

which implies \( \frac{\log(R_p)}{\log d_p} > \frac{1}{2} \) for large \( p \), that is, \( d_p^{-\varepsilon} < (R_p)^{-2\varepsilon} \). So we have

\[
\nu(B(x, r)) = \nu(B(x, r) \cap K) = \mu(\bigcup_{\omega \in \tilde{\Omega}^p} [\omega]) \\
\leq \#\tilde{\Omega}^p_{x, r} \cdot N^{-p} \leq (1 + M_D)^n \cdot (d_p^{-\varepsilon}) \cdot (N N^{-p+1}) \\
\leq (1 + M_D)^n N R_{p+1}^{-2\varepsilon} \\
\leq (1 + M_D)^n N R_{p+1}^{-3\varepsilon}.
\]

(ii) If \( d_p \geq R_p \), by Corollary 2.1, we have \( C_{p, r} \leq (2 + M_D)^n \). Thus

\[
\nu(B(x, r)) = \nu(B(x, r) \cap K) = \mu(\bigcup_{\omega \in \tilde{\Omega}^p} [\omega]) \leq \#\tilde{\Omega}^p_{x, r} \cdot N^{-p} \\
\leq (2 + M_D)^n N R_{p+1}^{-2\varepsilon} \leq (2 + M_D)^n N R_{p+1}^{-3\varepsilon}.
\]

Take \( C = (2 + M_D)^n N \), then

\[
\nu(B(x, r)) \leq C r^{-3\varepsilon}.
\]

By the principle of mass distribution, we have \( \dim_H K \geq s - 3\varepsilon \), which yields \( \dim_H K \geq s \).

By (1), \( \dim_H K = s \), the proof is completed.

**Corollary 2.3** If \( \sup \frac{R_p}{d_p} < +\infty \), then \( \dim_H K = \lim \inf_{k \to \infty} \frac{k \log N}{\log R_k} \).

**Proof.** It is clear that \( \sup \frac{R_p}{d_p} < +\infty \) implies \( \lim_{p \to \infty} \frac{\log(d_p + R_p)}{\log d_p} = 1 \) which completes the proof. \( \blacksquare \)

**Corollary 2.4** Let \( D = \{0, I_n\} \) where \( I_n \) is \( n \times n \) identity matrix. If \( |a_p| \geq R_p \) for large \( p \), then \( \dim_H K = \lim \inf_{k \to \infty} \frac{k \log 2}{\log R_k} \).

**Proof.** For any distinct \( \sigma, \tau \in \Omega^p \), we have

\[
|\pi(\sigma * 0^\infty) - \pi(\sigma * 0^\infty)| = \left| \sum_{i=1}^p E_i a_i \right|
\]

where \( E_i \in \{0, -I_n, I_n\} \). Let \( k = \min\{i : \sigma_i \neq \tau_i, 1 \leq i \leq p\} \), thus

\[
|\pi(\sigma * 0^\infty) - \pi(\sigma * 0^\infty)| \\
\geq |a_k| - \left| \sum_{i=k+1}^p E_i a_i \right| \\
\geq |a_k| - \sum_{i=k+1}^p |a_i| \\
= |a_k| - R_k + R_p.
\]

Therefore, \( 2d_p \geq R_p \).

By Corollary 2.3, \( \dim_H K = \lim \inf_{k \to \infty} \frac{k \log 2}{\log R_k} \). \( \blacksquare \)

**Remark 2.1** Corollary 2.4 implies Theorem 3 in [6] where the condition is \( \sup \frac{R_p}{|a_p|} < 1 \).

IJNS homepage: http://www.nonlinearscience.org.uk/
3 Some examples

Example 3.1 Take the series \( \sum_{k=1}^{\infty} \frac{2}{3^k} \), \( D = \{0, 1\} \), then the corresponding fractal is the classical middle-third Cantor set. Since \( a_k = \frac{2}{3^k} > \frac{1}{3^k} = R_k \), by Corollary 2.4, we have

\[
\dim_H K = \liminf_{k \to 0} \frac{k \log 2}{-\log R_k} = \frac{\log 2}{\log 3}.
\]

Example 3.2 Take the series \( \sum_{k=1}^{\infty} \frac{1}{3^k} \), \( D = \{0, \lambda, 2\}, \lambda \in [0, 1] \), then the corresponding fractal is the \( \lambda \)-Cantor set.

Let \( \frac{q}{m} \in \mathbb{Q} \) as in [11], then \( m(D - D) \subset \mathbb{Z} \) and \( d_p \geq 3^{-p} > 0 \) for all \( p \in \mathbb{N} \). Since \( R_p \leq \frac{1}{3^p} \), it follows that \( R_p d_p \leq 2m \). By Corollary 2.3, we have

\[
\dim_H K = \liminf_{k \to 0} \frac{k \log 3}{-\log R_k} = \frac{\log 3}{\log 3} = 1.
\]

Example 3.3 Take the series \( \sum_{k=1}^{\infty} a_k \) with \( a_k = \left( \begin{array}{c} \frac{1}{2} \\
0 \end{array} \right) \in \mathbb{R}^2 \) and

\[
D = \left\{ \left( \begin{array}{c} 0 \\
0 \end{array} \right), \left( \begin{array}{c} 1 \\
0 \end{array} \right), \left( \begin{array}{c} \frac{1}{2} \\
\frac{\sqrt{3}}{2} \end{array} \right), \left( \begin{array}{c} \frac{1}{2} \\
\frac{\sqrt{3}}{2} \end{array} \right) \right\},
\]

then the corresponding fractal is the Sierpinski triangle or gasket. It is easy to check that \( d_p \geq \frac{1}{2} (\frac{1}{2^p} + \frac{1}{2}) \) and \( R_p = \frac{1}{2^p} \), which yields \( \frac{R_p}{d_p} \leq 2 \). By Corollary 2.3,

\[
\dim_H K = \liminf_{k \to 0} \frac{k \log 3}{-\log R_k} = \frac{\log 3}{\log 2}.
\]

Example 3.4 Take the series \( \sum_{k=1}^{\infty} a_k \) with \( a_1 = \left( \begin{array}{c} 0 \\
0 \end{array} \right) \) and \( a_k = \left( \begin{array}{c} \frac{1}{4} \\
\prod_{i=2}^{k} (1 - \frac{1}{i}) \end{array} \right) \) for \( k \geq 2 \),

\[
D = \left\{ \left( \begin{array}{c} 0 \\
0 \end{array} \right), \left( \begin{array}{c} 2 \\
0 \end{array} \right), \left( \begin{array}{c} 0 \\
2 \end{array} \right), \left( \begin{array}{c} 2 \\
2 \end{array} \right) \right\}.
\]

It is easy to check that \( R_p = \frac{1}{4} \left( \frac{1}{2^p} \right) \) and

\[
c_1 \left( \frac{1}{4} \prod_{i=2}^{p} \left( 1 - \frac{1}{i} \right) \right) \leq d_p \leq c_2 \left( \frac{1}{4} \prod_{i=2}^{p} \left( 1 - \frac{1}{i} \right) \right)
\]

where \( c_1, c_2 \) are positive constants. Thus \( \lim_{p \to 0} \frac{\log R_p}{\log d_p} = 1 \) which yields \( \lim_{p \to 0} \frac{\log(d_p + R_p)}{\log d_p} = 1 \). By Theorem 1.1, the Hausdorff dimension of the corresponding fractal \( K \) is

\[
\dim_H K = \liminf_{p \to 0} \frac{p \log 4}{-\log R_p} = \frac{\log 4}{\log 4} = 1.
\]

However, \( \frac{R_p}{d_p} \to \infty \) as \( p \to \infty \) which illustrate that the condition of Corollary 2.3 is stronger than that of Theorem 1.1.

Acknowledgements

The author will thank Prof. Zhi-Xiong WEN for his helpful comments. This work was supported by Natural Science Foundation of Ningbo (Grant No. 2009A610077).

References


