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Stability of a Time-delay and Impulsive Financial Model

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Abstract: For an advertising impulsive method, time delay is inevitable at the impulsive moments. The stability for a financial model with time-delayed impulses is investigated. A sufficient condition is obtained to guarantee the equilibrium point to be globally asymptotically stable. The condition reflects the relationship between the system parameters, impulse intervals and time delays.The result is illustrated to be efficient through emulation, and presents economics explanation.

Keywords: financial model; time-delayed impulse; equilibrium point; globally asymptotically stable

1 Introduction

With the development of market economy, competition becomes more and more fierce. if we want to introduce a new product into a market, we have to advertise it in order to persuade consumers to purchase. Advertising effect is restricted by all kinds of factors, for example society, economy, culture and region. Therefore, advertising effect is lag, which is time delay. Time delays make system complicated , and have influence to the stability. The theory of delay differential systems has attracted many authors' attentions and significant progress has been made. Time delays often occur in biological, economic, neural networks and chemical systems [1-10].

Huller [11] presented a financial model which deals with introducing a new product into a market of size N. Sun[12] studied the model by using theory of impulsive control, and obtained some results. Time delay is a inevitable phenomenon. So when time delays exist between advertising investment and product sale. How often should we advertise the new product in order to attract the biggest number of consumers? The answer of this question makes a significant sense when advertisements become very expensive.

In this paper, we add the factor of time delay into model [12]. This paper is organized as follows. Some preliminaries are provided in Section 2 . In Section 3, a new theorem on the stability of a time-delay and impulsive financial model are presented. In Section 4, two simulation examples are used to show the effectiveness of the obtained results.

2 Preliminaries

A time-delay and impulsive differential system

$$\begin{cases} \dot{X}(t) = AX(t) + \Phi(x(t)), & t \neq t_k, k = 1, 2, \cdots \\ \triangle X(t) = BX(t - \tau_k), & t = t_k \end{cases} \tag{1}$$

where $\Phi : R^+ \times R^n \longrightarrow R^n$ is continuous, $X \in R^n$ is the state variable; $0 < t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots, t_k \to \infty$, as $k \to \infty, \delta_k = t_k - t_{k-1} < \infty, k = 1, 2, \cdots$ and $0 < \tau_k < \delta_k$.

Definition 1 [5] *: Let $V : R^+ \times R^n \longrightarrow R^+$, then V is said to belong to class V_0 if*
(1)V is continuous in $(t_{k-1}, t_k] \times R_n$ and for each $X \in R^n, k = 1, 2, \cdots, \lim_{(t,Y)\to(t_k^+,\ X)} V(t, Y) = V(t_k^+, X)$ exists;
(2) V is locally Lipschitzian in X.

Definition 2 [5] *: Let $V \in V_0$, for $(t, x) \in (t_{k-1}, t_k] \times R_n$, we define $D^+V(t, x(t)) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t + h, X(t + h)) - V(t, X(t))]$*

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3 Stability of a financial model

In this section, we study a financial model with time-delay and impulse. The financial model which deals with introducing a new product into a market of size N. N consists of the following three components:

(1) $x(t)-$the number of people who are unaware of the existence of the product;

(2) $y(t)-$the number of potential customers who are aware of the product but have not get purchased it,

(3) $z(t)-$the number of current customers who have purchased the product.

We can easily see that $x(t)+y(t)+z(t)=N$.

Advertising consists of two components:

(1)Awareness u,which informs consumers about the product and thus transfers them from the unaware group x into the potential group y,

(2)Trial advertising v,which persuades consumers to purchase the product and transfers them from the potential consumers y into the current consumers z.

The advertisement broadcast through a network, a national wide TV network, and so on. But between advertising investment and product sale have time delay. So we can model the advertising effect by time-delay and impulse. In this sense, the flows of consumers out and into different groups are given in the following impulsive model:

$$\begin{cases} \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{pmatrix} = \begin{pmatrix} -k & 0 & 0 \\ k & -a & \delta \\ 0 & a & -\delta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} + \begin{pmatrix} \frac{k}{N} \\ -\frac{k}{N} \\ 0 \end{pmatrix} X_1^2 \\ \Delta \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} -k_u & 0 & 0 \\ k_u & -k_v & 0 \\ 0 & k_v & 0 \end{pmatrix} \begin{pmatrix} X_1(t-\tau_k) \\ X_2(t-\tau_k) \\ X_3(t-\tau_k) \end{pmatrix} \end{cases}$$

where $k<0$ is the contact rate, $a>0$ is the trial rate, $\delta>0$ is the switching rate. k_u and k_v are the awareness rate and trial rate, respectively.

The system in (2) has two equilibrium points$(N,0,0)$ and $(0,\frac{\delta}{a+\delta}N,\frac{\delta}{a+\delta}N)$. In view of $k<0, a>0$ and $\delta>0$,we find that $(0,\frac{\delta}{a+\delta}N,\frac{\delta}{a+\delta}N)$ is an unstable equilibrium point, the equilibrium point$(N,0,0)$has not practical meaning. Our objective is to make the system to the equilibrium point $(0,\frac{\delta}{a+\delta}N,\frac{\delta}{a+\delta}N)$.

Let $X=(X_1,X_2,X_3)^T$, then we can rewrite the system (2) into the form

$$\begin{cases} \dot{X}(t) = AX(t) + \Phi(x(t)), & t \neq t_k, k=1,2,\cdots \\ \triangle X(t) = BX(t-\tau_k), & t=t_k \end{cases} \tag{2}$$

where $A=\begin{pmatrix} -k & 0 & 0 \\ k & -a & \delta \\ 0 & a & -\delta \end{pmatrix}$, $\Phi(X(t))=\begin{pmatrix} \frac{k}{N} \\ -\frac{k}{N} \\ 0 \end{pmatrix} X_1^2$, $B=\begin{pmatrix} -k_u & 0 & 0 \\ k_u & -k_v & 0 \\ 0 & k_v & 0 \end{pmatrix}$

Note that $|X_1(t)|\leq N, -\frac{\delta}{a+\delta}N\leq X_2(t)\leq\frac{\delta}{a+\delta}N, -\frac{\delta}{a+\delta}N\leq X_3(t)\leq\frac{\delta}{a+\delta}N$

We give a theorem to guarantee that the delayed and impulsive system will be stabilized at the desired equilibrium point.

Theorem 1 *Let λ_0 is the maximal eigenvalue of matrix $\frac{1}{2}(A^T+A)$. If there is a constant $\gamma<1$ such that $[\parallel I+B \parallel +\tau_k \parallel B \parallel (\parallel A \parallel +\sqrt{2})\frac{k}{N}]\exp(\lambda\delta_k)\leq\gamma, k=1,2,\cdots$ holds, then the equilibrium point of system (3) is globally asymptotically stable, where $\lambda=\lambda_0+k(1+\frac{\delta}{a+\delta})$*

Proof. Let the Lyapunov function be in the form of $V(X)=\parallel X(t)\parallel=(X^TX)^{\frac{1}{2}}$.

The right upper derivative of $V(X)$ along the solution of (3) is

$$\begin{aligned} D^+V(X(t)) &= \frac{1}{2\|X\|}[(AX+\Phi(X))^TX+X^T(AX+\Phi(X))] \\ &= \frac{1}{2\|X\|}[X^T(A^T+A)X+\Phi^T(X)X+X^T\Phi(X)], \end{aligned}$$

where $X^T(A^T+A)X\leq 2\lambda_0\parallel X(t)\parallel^2$.

$$\begin{aligned} \Phi^T(X)X+X^T\Phi(X) &= 2\frac{k}{N}X_1^2(X_1-X_2) \\ &= 2\frac{k}{N}[X_1X^T\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}X - X_2(t)X^T\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}X] \\ &\leq 2k(1+\frac{\delta}{a+\delta})\parallel X(t)\parallel^2 \end{aligned}$$

Thus $D^+V(X(t)) \leq \lambda \parallel X(t) \parallel, \quad t \in (t_{k-1}, t_k]$, which implies that

$$\parallel X(t) \parallel \leq \parallel X(t_{k-1}^+) \parallel \exp[\lambda(t - t_{k-1})] \tag{3}$$

From the second equation of system (3), we have

$$\begin{aligned} X(t_k^+) &= X(t_k) + BX(t_k - \tau_k) \\ &= (I + B)X(t_k) + B[X(t_k - \tau_k) - X(t_k)] \\ &= (I + B)X(t_k) - \tau_k B_k \dot{X}(t), \quad t \in (t_k - \tau_k, t_k] \end{aligned} \tag{4}$$

Assume $0 \leq \tau_k \leq \delta_k$,thus $t \in (t_k - \delta_k, t_k] \subset t \in (t_{k-1}, t_k]$

From (5), we have

$$\begin{aligned} \parallel X(t_k^+) \parallel &\leq \parallel I + B \parallel \parallel X(t_k) \parallel + \tau_k \parallel B \parallel \parallel \dot{X}(t) \parallel \\ &\leq \parallel I + B \parallel \parallel X(t_k) \parallel + \tau_k \parallel B \parallel (\parallel A \parallel + \sqrt{2}\tfrac{k}{N}) \parallel X(t) \parallel, \ \ t \in (t_{k-1}, t_k] \end{aligned} \tag{5}$$

In(4), we make $t = t_k$, then

$$\begin{aligned} \parallel X(t_k) \parallel &\leq \parallel X(t_{k-1}^+) \parallel \exp[\lambda(t_k - t_{k-1})] \\ &= \parallel X(t_{k-1}^+) \parallel \exp(\lambda\delta_k) \end{aligned} \tag{6}$$

From (7) , (6) change into the form of

$\parallel X(t_k^+) \parallel \leq [\parallel I + B \parallel + \tau_k \parallel B \parallel (\parallel A \parallel + \sqrt{2}\frac{k}{N})] \exp(\lambda\delta_k) \parallel X(_{k-1}^+) \parallel$

According to the condition of theorem 3.1, we have

$\parallel X(t_k^+) \parallel \leq \gamma \parallel X(t_{k-1}^+) \parallel \leq \gamma^2 \parallel X(t_{k-2}) \parallel \cdots \leq \gamma^k \parallel X(t_0) \parallel$

For $t \in (t_k, t_{k+1}]$,

$\lim_{k\to\infty} \parallel X(t) \parallel \leq \lim_{k\to\infty}[\parallel X(t_k^+) \parallel \exp(\lambda\delta_{k+1})] \leq \lim_{k\to\infty}[\gamma^k \parallel X(t_0) \parallel \exp(\lambda\delta_{k+1})] = 0$

Therefore, the equilibrium point of system (3) is globally asymptotically stable. ■

Especially, when $\tau_k = 0, k = 1, 2, \cdots$, the system (3) changes into the impulsive model of [6]

$$\begin{cases} \dot{X}(t) = AX(t) + \Phi(x(t)), \quad t \neq t_k, k = 1, 2, \cdots \\ \triangle X(t) = BX(t), \quad t = t_k \end{cases} \tag{7}$$

We have the following corollary:

Corollary 2 *Let λ_0 is the maximal eigenvalue of matrix $\frac{1}{2}(A^T + A)$. If there is a constant $\gamma < 1$ such that $\parallel I + B \parallel \exp(\lambda\delta_k) \leq \gamma, k = 1, 2, \cdots$ holds, then the equilibrium point of system (8) is globally asymptotically stable. where $\lambda = \lambda_0 + k(1 + \frac{\delta}{a+\delta})$*

Remark 3 *The condition of theorem 3.1 has significant real meaning. For $\forall\ \gamma < 1$, if impulsive interval δ_k is long , the time delay τ_k is short. Therefore, if we want to introduce a new product into a market through advertisement, the advertisement must have quick effect in order to reduce the number of advert (i.e., decrease cost).*

4 Example

Consider the system (3), where $a = 8, k = -1, \delta = 25, N = 10, \tau_k = 0.1, \delta_k = 1$,

$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -8 & 25 \\ 0 & 8 & -25 \end{pmatrix}, B = \begin{pmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.6 & 0 \\ 0 & -0.6 & 0 \end{pmatrix}$

then $\parallel I + B \parallel = 1.1917, \parallel B \parallel = 0.9641, \parallel A \parallel = 1.1741, \lambda = -0.4539$

For $\gamma < 1$, such that $[\parallel I + B \parallel + \tau_k \parallel B \parallel (\parallel A \parallel + \sqrt{2})\frac{k}{N}] \exp(\lambda\delta_k) \leq \gamma$ holds.

We know that the equilibrium point $(0, \frac{\delta}{a+\delta}N, \frac{\delta}{a+\delta}N)$ of the impulsive controlled financial model with above parameters is asymptotically stable.

5 Conclusion

In this paper, we set up a time-delay and impulsive financial model. A sufficient condition is obtained to guarantee the equilibrium point to be globally asymptotically stable. The condition reflects the relationship between the system parameters, impulse intervals and time delays , and has some real meaning. To some extent , the strategy of advertising investment have scientific reference. In order to decrease the difference between model and reality, we will further study the problem.

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