Simultaneous Accumulation Points to Sets of $d$-tuples

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Abstract: In this paper, we consider a class of fractals associated with accumulation points in $d$-tuples. By constructing some homogeneous Moran subset, we prove that these fractals have full dimension.

Keywords: accumulation point; Hausdorff dimension; homogeneous Moran set

1 Introduction

Recently, full dimension subsets have attracted much interest in literature. Some subsets of divergence points, self-similar set, finite Graph and symbolic system [1-4] may have full dimension. In this paper, we consider a class of fractals associated with accumulation points and their Hausdorff dimension, we extend the result of [5].

Cantor series expansion is one of the most applied operation of mathematics in representations of real numbers. We first introduce some definitions on Cantor series expansion. Let $Q = \{q_k\}$ be a sequence of integers with $q_k \geq 2$. Suppose $d_k(x) \in \mathbb{Z} \cap [0, q_k - 1]$ for all $k$, we call the representation $x = \sum_{k \geq 1} \frac{d_k(x)}{q_1 q_2 \cdots q_k}$ \hspace{1cm} (1.1)
the Cantor series of $x$ with respect to $Q$, if $d_k(x)/q_k \neq 0$ for infinitely many values of $k$. We call the integers $\{d_k(x)\}_{k=1}^{\infty}$ the digits of this expansion.

By [6], if $\limsup_{k \to \infty} q_k = +\infty$, then for almost all $x \in [0, 1]$, we have
\[ \limsup_{k \to \infty} \frac{d_k(x)}{q_k} = 1 \quad \text{and} \quad \liminf_{k \to \infty} \frac{d_k(x)}{q_k} = 0. \]

Before we state our main theorem, let us recall some notations. For a $d$ ($d$ is a positive integer equal or larger than 1) tuple of $x = (x_1, \cdots, x_d) \in [0, 1]^d$ and a positive integer $i$, let $d_i(x) = (d_i(x_1), \cdots, d_i(x_d))$
i.e. $d_i(x)$ is the vector consisting of the $i$-th digits of $x_j$’s ($j = 1, \cdots, d$). Next, write $\Sigma_{q_k} = \{0, 1, \cdots, q_k - 1\}^d$
i.e. $\Sigma_{q_k}$ is the family of vector $i = (i_1, \cdots, i_d)$ with entries $i_j \in \{0, 1, \cdots, q_k - 1\}$ and for a positive integer $k$, write $\Sigma_{q_1 \cdots q_k} = \Sigma_{q_1} \times \cdots \times \Sigma_{q_k}$
for the family of strings of $\omega = i_1 \cdots i_k$ of length $k$ whose entries $i_j \in \Sigma_{q_j}$ are vectors of Cantor expansion digits. For $x = (x_1, \cdots, x_d) \in [0, 1]^d$ and the integer $k$, we let $d_i(x) \cdots d_{i+k-1}(x)$

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denote the string in \(\Sigma_{q_1\cdots q_{k+1}}\) obtained by concatenating the following \(k\) vectors: \(d_i(x), \cdots, d_{i+k-1}(x)\).

Let \(A\left(\left\{\frac{d_k(x)}{q_k}\right\}_{k=1}^{\infty}\right)\) denote the set of all accumulation points of \(\left\{\frac{d_k(x)}{q_k}\right\}_{k=1}^{\infty}\). For a closed set \(A \subseteq [0,1]^d\) and \(k = 1, 2, \cdots\), let

\[
E_A = \left\{ x \in [0,1]^d : x = \sum_{k \geq 1} \frac{d_k(x)}{q_1q_2\cdots q_k} \text{ with } d_k(x_i) \in \mathbb{Z} \cap [0,q_k-1] \text{ and } A\left(\left\{\frac{d_k(x)}{q_k}\right\}_{k=1}^{\infty}\right) = A \right\}.
\]

Now, we are in the position to state our main result.

**Theorem 1** If \(\lim_{k \to \infty} q_k = +\infty\), then \(\dim_H E_A = d\) for every closed set \(A \subseteq [0,1]^d\), where \(\dim_H(\cdot)\) denote the Hausdorff dimension [7,8].

The technique of this paper is to construct a Moran subset of \(E_A\) such that this Moran subset has full dimension \(d\), the method is referred from [5].

# 2 Proof of theorem

## 2.1 Homogeneous Moran set

Let \(\{n_k\}_{k=1}^{\infty}\) be a sequence of positive integers and \(\{c_k\}_{k=1}^{\infty}\) a sequence of positive numbers satisfying \(n_k \geq 2, 0 < c_k < 1, n_1c_1 < \eta\) and \(n_kc_k \leq 1 (k \geq 2)\), where \(\eta\) is a positive number.

For any \(k \geq 1\), let \(D_k = \{(i_1, \cdots, i_k) : 1 \leq i_j \leq n_j, 1 \leq j \leq k\}\), \(D = \bigcup_{k \geq 0} D_k\) and \(D_0 = \emptyset\). If \(\sigma = (i_1, \cdots, i_k) \in D_k\), write \(\sigma \ast \tau = (i_1, \cdots, i_k, j_1, \cdots, j_m)\).

Suppose that \(J\) is a closed unit cube in \(\mathbb{R}^d\), the collection of subset \(\mathcal{F} = \{\mathcal{J}_\sigma : \sigma \in D\}\) of \(J\) has a homogeneous Moran structure, if

1. \(J_0 = J\);
2. For every \(k \geq 0\) and \(\sigma \in D_k\), \(J_\sigma \ast 1, \cdots, J_\sigma \ast (n_k + 1)\) are subset of \(J_\sigma\) with their interiors pairwise disjoint;
3. For every \(k \geq 1\) and any \(\sigma \in D_{k-1}\), \(1 \leq j \leq n_k\), we have \(c_k = \frac{|J_\sigma|}{|J_{\sigma \ast j}|}\), where \(|J_\sigma|\) denote the diameter of \(J_\sigma\).

We call \(E(\mathcal{F}) \triangleq \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} J_\sigma\) a high dimensional homogeneous Moran set determined by \((J, \{n_k\}^d, \{c_k\})\), related information can refer to [9]. Let \(\mathcal{M} = \mathcal{M}(J, \{n_k\}^d, \{c_k\})\) be the collection of homogeneous Moran set determined by \((J, \{n_k\}^d, \{c_k\})\).

**Lemma 2** [9] Suppose \(E \in \mathcal{M} = \mathcal{M}(J, \{n_k\}^d, \{c_k\})\), then

\[
\dim_H E \geq \liminf_{k \to \infty} \frac{d \log_2(n_1 \cdots n_k)}{-\log_2(c_1 \cdots c_k c_{k+1} n_{k+1})}.
\]

## 2.2 Proof of theorem

Our main idea is to construct a homogeneous Moran subset of \(E_A\) with Hausdorff dimension \(d\).

Fixing the sequence \(\{q_k\}_{k=1}^{\infty}\), let

\[
\varepsilon_k = \min\left\{\left(\log_2 q_k\right)^2, \left[\log_2(1 \cdots q_k)\right]^2\right\} \frac{1}{\log_2 q_k},
\]

then we can have

\[
\lim_{k \to \infty} \varepsilon_k = 0,
\]

since \(\varepsilon_k \leq (\log_2 q_k)^2 = \frac{1}{\log_2 q_k} \to 0\) as \(k \to \infty\).

By the assumption of Theorem 1, we have \(\lim_{k \to \infty} q_k = +\infty\) and \(\lim_{k \to \infty} (q_1 \cdots q_{k-1}) \geq \lim_{k \to \infty} 2^{k-1} = +\infty\). Thus

\[
\lim_{k \to \infty} q_k^{-\varepsilon_k} = 0.
\]
Given $\delta \in [0, 1]$, we define a sequence $\{I_{\delta,k}\}$ of intervals as follows

$$I_{\delta,k} = \begin{cases} \left[ \delta q_k - q_k^{1-\varepsilon_k} - 2, \delta q_k - 1 \right], & \text{if } \delta \in \left[ \frac{1}{2}, 1 \right]; \\ \left[ \delta q_k, \delta q_k + q_k^{1-\varepsilon_k} + 1 \right], & \text{if } \delta \in \left[ 0, \frac{1}{2} \right]. \end{cases}$$

(2.4)

**Lemma 3** [5] Suppose $\lim_{k \to \infty} q_k = +\infty$, and $\{\varepsilon_k\}_k, \{I_{\delta,k}\}_k$ are defined as above. There exists some integer $k_0 > 0$ such that for any $\delta \in [0, 1]$ and any $k \geq k_0$, we have

$$I_{\delta,k} \subset [0, q_k - 1] \text{ and } \mathcal{J}(I_{\delta,k} \cap \mathbb{Z}) \geq q_k^{1-\varepsilon_k} \geq 2.$$  

(2.5)

Let $a_k = (a_{k,1}, \ldots, a_{k,d}), b_k = (b_{k,1}, \ldots, b_{k,d})$.

We choose a denumerable dense subset $\Delta = \{a_1, \ldots, a_k, \ldots\}$ of $\mathbb{A}$ and let $\{b_n\}_{n \geq 1}$ denote the sequence $\{a_1, a_1, a_2, a_1, a_2, a_3, a_1, a_2, a_3, a_4, \ldots\}$, i.e. for any positive integer $k$

$$b_{\frac{k(k-1)}{2} + j} = a_j$$

(2.6)

for any $j \in \mathbb{N} \cap [1, k]$.

Suppose $k_0$ is the integer mentioned in Lemma 2. Let

$$M_{A} = \left\{ x \in [0,1]^d : x = \sum_{k \geq 1} \frac{d_k(x)}{q_k} \right\} \cap [0, q_k - 1] \cap I_{bk,i} \forall k \geq k_0 \right\}.$$  

Then we have

**Lemma 4** $M_{A} \subseteq E_{A}$.

**Proof.** It suffices to prove that for any $x = (x_1, \ldots, x_d) \in M_{A}$,

$$A\left( \left\{ \frac{d_k(x)}{q_k} \right\}^\infty_{k=1} \right) = A.$$  

(2.7)

Given $x \in M_{A}$, by (2.4) the following estimation holds for each $k \geq 1$:

$$\left| \frac{d_k(x)}{q_k}, b_k \right| = \sqrt{ \left( \frac{d_k(x_1)}{q_k} - b_{1,k} \right)^2 + \cdots + \left( \frac{d_k(x_d)}{q_k} - b_{d,k} \right)^2 } \leq \sqrt{d} (q_k^{-1} + q_k^{-\varepsilon_k}).$$

(2.8)

For any $a_l \in \Delta$, we can select a subsequence $\{n_k^{(l)}\}_{k \geq 1}$ such that $b_{n_k^{(l)}} = a_l$. Then by (2.8), the following holds

$$\lim_{k \to \infty} \frac{d_k^{(l)}(x_1)}{q_k} = a_{l,1}, \ldots, \lim_{k \to \infty} \frac{d_k^{(l)}(x_d)}{q_k} = a_{l,d}.$$  

Because $\Delta$ is a dense subset of $A$, we have

$$A\left( \left\{ \frac{d_k(x)}{q_k} \right\}^\infty_{k=1} \right) \supseteq A.$$  

(2.9)

On the other hand, if $x^* = (x_1^*, \ldots, x_d^*) \in A\left( \left\{ \frac{d_k(x)}{q_k} \right\}^\infty_{k=1} \right)$, then there exist a subsequence $\{m_k\}_k$ such that

$$\lim_{k \to \infty} \frac{d_k(m_k(x_1))}{q_k} = x_1^*, \ldots, \lim_{k \to \infty} \frac{d_k(m_k(x_d))}{q_k} = x_d^*.$$  

Since $A$ is a closed set and

$$\left| \frac{d_k(m_k(x))}{q_k}, b_{m_k} \right| \leq \sqrt{d} (q_k^{-1} + q_k^{-\varepsilon_k}) \to 0 \text{ as } k \to \infty,$$

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we deduce
\[
\mathbf{x}^* = \left( \lim_{k \to \infty} \frac{d_{m_k}(x_1)}{q_k}, \ldots, \lim_{k \to \infty} \frac{d_{m_k}(x_d)}{q_k} \right) \\
= \left( \lim_{k \to \infty} b_{m_{k,1}}, \ldots, \lim_{k \to \infty} b_{m_{k,d}} \right) \\
\in A.
\] (2.10)

which implies
\[
\mathbb{A} \left( \left\{ \frac{d_k(x)}{q_k} \right\}_{k=1}^\infty \right) \subseteq A.
\] (2.11)

By (2.9) and (2.11), we complete the proof. ■

**Lemma 5** \( \dim_H M_A = d \)

**Proof.** The set \( M_A \) is a homogeneous Moran set with \( c_k = q_k^{-1} \). By Lemma 2, for any \( k \geq k_0 \) and \( b_{k,i} \in [0, 1] \),
\[
n_k = \#(\mathbb{Z} \cap [0, q_k - 1] \cap I_{b_{k,i}}) \geq q_k^{1-\varepsilon_k} \geq 2.
\] (2.12)

And \( n_k = q_k(\geq 2) \) for any \( k < k_0 \).

By (2.1), one can easily see
\[
\lim_{k \to \infty} \frac{\varepsilon_k \log_2 q_k}{\log_2 (q_1 \cdots q_{k-1})} = 0.
\] (2.13)

Using Lemma 1, (2.2), (2.12) and (2.13), we have
\[
\dim_H M_A \geq \liminf_{k \to \infty} \frac{d \log_2 (n_1 \cdots n_k)}{- \log_2 (c_1 \cdots c_k c_{k+1} n_{k+1})} \\
\geq \liminf_{k \to \infty} \frac{d \left[ \log_2 (q_1 \cdots q_{k_0}) + \log_2 (q_{k_0+1}^{1-\varepsilon_{k_0+1}} \cdots q_k^{1-\varepsilon_k}) \right]}{- \log_2 \left( \frac{1}{q_1} \cdots \frac{1}{q_{k+1}} \left( q_{k+1}^{1-\varepsilon_{k+1}} \right) \right)} \\
= \lim_{k \to \infty} \frac{d \log_2 (q_1 \cdots q_k)}{\log_2 (q_1 \cdots q_k) + \varepsilon_{k+1} \log_2 q_{k+1}} \\
= \lim_{k \to \infty} \left( \frac{d}{1 + \varepsilon_{k+1} \log_2 q_{k+1}} \right) \\
= d
\]

**Proof of Theorem 1.** The conclusion of Theorem 1 follows immediately by Lemma 3 and Lemma 4. ■

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**References**


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