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# Simultaneous Accumulation Points to Sets of $d$-tuples

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**Abstract:** In this paper, we consider a class of fractals associated with accumulation points in $d$-tuples. By constructing some homogeneous Moran subset, we prove that these fractals have full dimension.

**Keywords:** accumulation point; Hausdorff dimension; homogeneous Moran set

## 1 Introduction

Recently, full dimension subsets have attracted much interest in literature. Some subsets of divergence points, self-similar set, finite Graph and symbolic system [1-4] may have full dimension. In this paper, we consider a class of fractals associated with accumulation points and their Hausdorff dimension, we extend the result of [5].

Cantor series expansion is one of the most applied operation of mathematics in representations of real numbers. We first introduce some definitions on Cantor series expansion. Let $Q = \{q_k\}$ be a sequence of integers with $q_k \geq 2$. Suppose $d_k(x) \in \mathbb{Z} \cap [0, q_k - 1]$ for all $k$, we call the representation

$$x = \sum_{k \geq 1} \frac{d_k(x)}{q_1 q_2 \cdots q_k} \tag{1.1}$$

the Cantor series of $x$ with respect to $Q$, if $d_k(x) \neq 0$ for infinitely many values of $k$. We call the integers $\{d_k(x)\}_{k=1}^{\infty}$ the digits of this expansion.

By [6], if $\limsup_{k\to\infty} q_k = +\infty$, then for almost all $x \in [0, 1]$, we have

$$\limsup_{k\to\infty} \frac{d_k(x)}{q_k} = 1 \text{ and } \liminf_{k\to\infty} \frac{d_k(x)}{q_k} = 0.$$

Before we state our main theorem, let us recall some notations. For a $d$ ($d$ is a positive integer equal or larger than 1) tuple of $\mathbf{x} = (x_1, \cdots, x_d) \in [0, 1]^d$ and a positive integer $i$, let

$$\mathbf{d}_i(\mathbf{x}) = (d_i(x_1), \cdots, d_i(x_d))$$

i.e. $\mathbf{d}_i(\mathbf{x})$ is the vector consisting of the $i$-th digits of $x_j$'s ($j = 1, \cdots, d$). Next, write

$$\Sigma_{q_k} = \{0, 1, \cdots, q_k - 1\}^d$$

i.e. $\Sigma_{q_k}$ is the family of vector $\mathbf{i} = (i_1, \cdots, i_d)$ with entries $i_j \in \{0, 1, \cdots, q_k - 1\}$ and for a positive integer $k$, write

$$\Sigma_{q_1 \cdots q_k} = \Sigma_{q_1} \times \cdots \times \Sigma_{q_k}$$

for the family of strings of $\omega = \mathbf{i}_1 \cdots \mathbf{i}_k$ of length $k$ whose entries $\mathbf{i}_j \in \Sigma_{q_j}$ are vectors of Cantor expansion digits. For $\mathbf{x} = (x_1, \cdots, x_d) \in [0, 1]^d$ and the integer $k$, we let

$$\mathbf{d}_i(\mathbf{x}) \cdots \mathbf{d}_{i+k-1}(\mathbf{x})$$

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denote the string in $\Sigma_{q_i \cdots q_{i+k-1}}$ obtained by concatenating the following $k$ vectors: $\mathbf{d}_i(\mathbf{x}), \cdots, \mathbf{d}_{i+k-1}(\mathbf{x})$.

Let $\mathbb{A}\Big(\{\frac{\mathbf{d}_k(\mathbf{x})}{q_k}\}_{k=1}^{\infty}\Big)$ denote the set of all accumulation points of $\{\frac{\mathbf{d}_k(\mathbf{x})}{q_k}\}_{k=1}^{\infty}$. For a closed set $A \subseteq [0,1]^d$ and $k = 1, 2, \cdots$, let

$$E_A = \Big\{\mathbf{x} \in [0,1]^d : \mathbf{x} = \sum_{k \geq 1} \frac{\mathbf{d}_k(\mathbf{x})}{q_1 q_2 \cdots q_k} \text{ with } d_k(x_i) \in \mathbb{Z} \cap [0, q_k - 1] \text{ and } \mathbb{A}\Big(\{\frac{\mathbf{d}_k(\mathbf{x})}{q_k}\}_{k=1}^{\infty}\Big) = A\Big\}.$$

Now, we are in the position to state our main result.

**Theorem 1** *If* $\lim_{k\to\infty} q_k = +\infty$, *then* $\dim_H E_A = d$ *for every closed set* $A \subseteq [0,1]^d$, *where* $\dim_H(.)$ *denote the Hausdorff dimension [7,8].*

The technique of this paper is to construct a Moran subset of $E_A$ such that this Moran subset has full dimension $d$, the method is referred from [5].

# 2 Proof of theorem

## 2.1 Homogeneous Moran set

Let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers and $\{c_k\}_{k\geq 1}$ a sequence of positive numbers satisfying $n_k \geq 2, 0 < c_k < 1, n_1 c_1 < \eta$ and $n_k c_k \leq 1 (k \geq 2)$, where $\eta$ is a positive number.

For any $k \geq 1$, let $D_k = \{(i_1, \cdots, i_k) : 1 \leq i_j \leq n_j, 1 \leq j \leq k\}$, $D = \cup_{k\geq 0} D_k$ and $D_0 = \emptyset$. If $\sigma = (i_1, \cdots, i_k) \in D_k, \tau = (j_1, \cdots, j_m) \in D_m$, write $\sigma * \tau = (i_1, \cdots, i_k, j_1, \cdots, j_m)$.

Suppose that $J$ is a closed unit cube in $\mathbb{R}^d$, the collection of subset $\mathcal{F} = \{J_\sigma : \sigma \in D\}$ of $J$ has a homogeneous Moran structure, if

(1) $J_\emptyset = J$;

(2) For every $k \geq 0$ and $\sigma \in D_k$, $J\sigma * 1, \cdots, J\sigma * (n_k + 1)$ are subset of $J_\sigma$ with their interiors pairwise disjoint;

(3) For every $k \geq 1$ and any $\sigma \in D_{k-1}, 1 \leq j \leq n_k$, we have $c_k = \frac{|J_{\sigma * j}|}{|J_\sigma|}$, where $|J_\sigma|$ denote the diameter of $J_\sigma$.

We call $E(\mathcal{F}) \triangleq \bigcap_{k\geq 1} \bigcup_{\sigma \in D_k} J_\sigma$ a high dimensional homogeneous Moran set determined by$(J, \{n_k\}^d, \{c_k\})$, related information can refer to [9]. Let $\mathcal{M} = \mathcal{M}(J, \{n_k\}^d, \{c_k\})$ be the collection of homogeneous Moran set determined by $(J, \{n_k\}^d, \{c_k\})$.

**Lemma 2** [9] *Suppose* $E \in \mathcal{M} = \mathcal{M}(J, \{n_k\}^d, \{c_k\})$, *then*

$$\dim_H E \geq \liminf_{k\to\infty} \frac{d \log_2(n_1 \cdots n_k)}{-\log_2(c_1 \cdots c_k c_{k+1} n_{k+1})}.$$

## 2.2 Proof of theorem

Our main idea is to construct a homogeneous Moran subset of $E_A$ with Hausdorff dimension $d$.

Fixing the sequence $\{q_k\}_{k=1}^{\infty}$, let

$$\varepsilon_k = \frac{\min\{(\log_2 q_k)^{\frac{1}{2}}, [\log_2(q_1 \cdots q_{k-1})]^{\frac{1}{2}}\}}{\log_2 q_k}, \tag{2.1}$$

then we can have

$$\lim_{k\to\infty} \varepsilon_k = 0, \tag{2.2}$$

since $\varepsilon_k \leq \frac{(\log_2 q_k)^{\frac{1}{2}}}{\log_2 q_k} = \frac{1}{(\log_2 q_k)^{\frac{1}{2}}} \to 0$ as $k \to \infty$.

By the assumption of Theorem 1, we have $\lim_{k\to\infty} q_k = +\infty$ and $\lim_{k\to\infty}(q_1 \cdots q_{k-1}) \geq \lim_{k\to\infty} 2^{k-1} = +\infty$. Thus

$$\lim_{k\to\infty} q_k^{-\varepsilon_k} = 0. \tag{2.3}$$

Given $\delta \in [0,1]$, we define a sequence $\{I_{\delta,k}\}$ of intervals as follows

$$I_{\delta,k} = \begin{cases} \left[\delta q_k - q_k^{1-\varepsilon_k} - 2, \delta q_k - 1\right], & \text{if } \delta \in [\frac{1}{2}, 1]; \\ \left[\delta q_k, \delta q_k + q_k^{1-\varepsilon_k} + 1\right], & \text{if } \delta \in [0, \frac{1}{2}). \end{cases} \tag{2.4}$$

**Lemma 3** [5] *Suppose* $\lim_{k\to\infty} q_k = +\infty$*, and* $\{\varepsilon_k\}_k, \{I_{\delta,k}\}_k$ *are defined as above. There exists some integer* $k_0 > 0$ *such that for any* $\delta \in [0,1]$ *and any* $k \geq k_0$*, we have*

$$I_{\delta,k} \subset [0, q_k - 1] \text{ and } \sharp(I_{\delta,k} \cap \mathbb{Z}) \geq q_k^{1-\varepsilon_k} \geq 2. \tag{2.5}$$

Let $\mathbf{a}_k = (a_{k,1}, \cdots, a_{k,d})$, $\mathbf{b}_k = (b_{k,1}, \cdots, b_{k,d})$.

We choose a denumerable dense subset $\triangle = \Big\{\mathbf{a}_1, \cdots, \mathbf{a}_k, \cdots\Big\}$ of $A$ and let $\{\mathbf{b}_n\}_{n\geq 1}$ denote the sequence $\Big\{\mathbf{a}_1, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \cdots\Big\}$. i.e. for any positive integer $k$

$$\mathbf{b}_{\frac{k(k-1)}{2}+j} = \mathbf{a}_j \tag{2.6}$$

for any $j \in \mathbb{N} \cap [1, k]$.

Suppose $k_0$ is the integer mentioned in Lemma 2. Let

$$M_A = \Big\{\mathbf{x} \in [0,1]^d : \mathbf{x} = \sum_{k\geq 1} \frac{\mathbf{d}_k(\mathbf{x})}{q_1 q_2 \cdots q_k} \text{ with } d_k(x_i) \in \mathbb{Z} \cap [0, q_k - 1] \cap I_{b_{k,i}} \text{ for all } k \geq k_0\Big\}.$$

Then we have

**Lemma 4** $M_A \subseteq E_A$.

**Proof.** It suffices to prove that for any $\mathbf{x} = (x_1, \cdots, x_d) \in M_A$,

$$\mathbb{A}\Big(\{\frac{\mathbf{d}_k(\mathbf{x})}{q_k}\}_{k=1}^{\infty}\Big) = A. \tag{2.7}$$

Given $\mathbf{x} \in M_A$, by (2.4) the following estimation holds for each $k \geq 1$:

$$\begin{aligned} \Big|\frac{\mathbf{d}_k(\mathbf{x})}{q_k}, \mathbf{b}_k\Big| &= \sqrt{(\frac{d_k(x_1)}{q_k} - b_{k,1})^2 + \cdots + (\frac{d_k(x_d)}{q_k} - b_{k,d})^2} \\ &\leq \sqrt{d}(q_k^{-1} + q_k^{-\varepsilon_k}). \end{aligned} \tag{2.8}$$

For any $\mathbf{a}_l \in \triangle$, we can select a subsequence $\{n_k^{(l)}\}_{k\geq 1}$ such that $\mathbf{b}_{n_k^{(l)}} = \mathbf{a}_l$. Then by (2.8), the following holds

$$\lim_{k\to\infty} \frac{d_{n_k}^{(l)}(x_1)}{q_k} = a_{l,1}, \cdots, \lim_{k\to\infty} \frac{d_{n_k}^{(l)}(x_d)}{q_k} = a_{l,d}.$$

Because $\triangle$ is a dense subset of $A$, we have

$$\mathbb{A}\Big(\{\frac{\mathbf{d}_k(\mathbf{x})}{q_k}\}_{k=1}^{\infty}\Big) \supseteq A \tag{2.9}$$

On the other hand, if $\mathbf{x}^* = (x_1^*, \cdots, x_d^*) \in \mathbb{A}\Big(\{\frac{\mathbf{d}_k(\mathbf{x})}{q_k}\}_{k=1}^{\infty}\Big)$, then there exist a subsequence $\{m_k\}_k$ such that

$$\lim_{k\to\infty} \frac{d_{m_k}(x_1)}{q_k} = x_1^*, \cdots, \lim_{k\to\infty} \frac{d_{m_k}(x_d)}{q_k} = x_d^*.$$

Since $A$ is a closed set and

$$\Big|\frac{\mathbf{d}_{m_k}(\mathbf{x})}{q_k}, \mathbf{b}_{m_k}\Big| \leq \sqrt{d}(q_k^{-1} + q_k^{-\varepsilon_k}) \to 0 \text{ as } k \to \infty,$$

we deduce

$$\begin{aligned}\mathbf{x}^* &= \Big(\lim_{k\to\infty}\frac{d_{m_k}(x_1)}{q_k},\cdots,\lim_{k\to\infty}\frac{d_{m_k}(x_d)}{q_k}\Big)\\ &= (\lim_{k\to\infty}b_{m_k,1},\cdots,\lim_{k\to\infty}b_{m_k,d})\\ &\in A. \end{aligned} \tag{2.10}$$

which implies

$$\mathbb{A}\Big(\{\frac{\mathbf{d}_k(\mathbf{x})}{q_k}\}_{k=1}^{\infty}\Big) \subseteq A. \tag{2.11}$$

By (2.9) and (2.11), we complete the proof. ■

**Lemma 5** $\dim_H M_A = d$

**Proof.** The set $M_A$ is a homogeneous Moran set with $c_k = q_k^{-1}$. By Lemma 2, for any $k \geq k_0$ and $b_{k,i} \in [0,1]$,

$$n_k = \sharp(\mathbb{Z}\cap[0,q_k-1]\cap I_{b_{k,i}}) \geq q_k^{1-\varepsilon_k} \geq 2. \tag{2.12}$$

And $n_k = q_k(\geq 2)$ for any $k < k_0$.

By (2.1), one can easily see

$$\lim_{k\to\infty}\frac{\varepsilon_k\log_2 q_k}{\log_2(q_1\cdots q_{k-1})} = 0. \tag{2.13}$$

Using Lemma 1, (2.2), (2.12) and (2.13), we have

$$\begin{aligned}\dim_H M_A &\geq \liminf_{k\to\infty}\frac{d\log_2(n_1\cdots n_k)}{-\log_2(c_1\cdots c_k c_{k+1}n_{k+1})}\\ &\geq \liminf_{k\to\infty}\frac{d[\log_2(q_1\cdots q_{k_0})+\log_2(q_{k_0+1}^{1-\varepsilon_{k_0+1}}\cdots q_k^{1-\varepsilon_k})]}{-\log_2[\frac{1}{q_1}\cdots\frac{1}{q_{k+1}}(q_{k+1}^{1-\varepsilon_{k+1}})]}\\ &= \lim_{k\to\infty}\frac{d\log_2(q_1\cdots q_k)}{\log_2(q_1\cdots q_k)+\varepsilon_{k+1}\log_2 q_{k+1}}\\ &= \lim_{k\to\infty}\frac{d}{1+\frac{\varepsilon_{k+1}\log_2 q_{k+1}}{\log_2(q_1\cdots q_k)}}\\ &= d\end{aligned}$$

■

**Proof of Theorem 1**. The conclusion of Theorem 1 follows immediately by Lemma 3 and Lemma 4.■

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