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Solving a Class of Two-dimensional Linear and Nonlinear Volterra Integral Equations by Means of the Homotopy Analysis Method

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Abstract: In this paper, a class of two-dimensional linear and nonlinear Volterra integral equations is solved by means of an analytic technique, namely the Homotopy analysis method (HAM). Comparisons are made between the differential transform method (DTM), the exact solution and the homotopy analysis method. The results reveal that the proposed method is very effective and simple.

Keywords: Volterra integral equations; Homotopy analysis method; Homotopy perturbation method;differential transform method

1 Introduction

In 1992, Liao[7] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely Homotopy analysis method (HAM)[7–10]. This method has been successfully applied to solve many types of nonlinear problems [5, 11–18]. The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy, and requires large computer power and time. HAM method is better since it does not involve discretization of the variables hence is free from rounding off errors and does not require large computer memory or time.

In this paper, we will consider the approximate solution of the linear and nonlinear Volterra integral equation of the second kind

$$u(x,t) - \int_0^t \int_0^x K(x,t,y,z,u(y,z))dydz = f(x,t), \tag{1}$$

where K and f are continuous functions and K has the following degenerate form

$$K(x,t,y,z) = \sum_{i=0}^{p} \nu_i(x,t)\omega_i(y,z,u(y,z)), \tag{2}$$

2 Basic idea of HAM

We apply the HAM to the Stefan problem with boundary and initial conditions. We consider the following differential equation

$$\mathcal{N}[u(\tau)] = 0, \tag{3}$$

where $\mathcal{N}$ is a nonlinear operator, τ denotes independent variable, $u(\tau)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [8] constructs the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] = p\,\hbar\mathcal{H}(\tau)\mathcal{N}[\phi(\tau;p)], \tag{4}$$

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where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $\mathcal{H}(\tau) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, $u(\tau;p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\phi(\tau;0) = u_0(\tau), \phi(\tau;1) = u(\tau), \tag{5}$$

respectively. Thus, as p increases from 0 to 1, the solution $u(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $u(\tau;p)$ in Taylor series with respect to p, we have

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m, \tag{6}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m}|_{p=0}. \tag{7}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (4) converges at $p = 1$, then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{8}$$

which must be one of solutions of original nonlinear equation, as proved by[8]. As $\hbar = -1$ and $\mathcal{H}(\tau) = 1$, Eq. (2) becomes

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] + p\,\mathcal{N}[\phi(\tau;p)] = 0, \tag{9}$$

which is used mostly in the homotopy perturbation method[2], where as the solution obtained directly, without using Taylor series [3, 4]. According to the definition (5), the governing equation can be deduced from the zero-order deformation equation (2). Define the vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\}.$$

Differentiating equation (2) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar\,\mathcal{H}(\tau)\Re_m(\vec{u}_{m-1}), \tag{10}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(\tau;p)]}{\partial p^{m-1}}|_{p=0}. \tag{11}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{12}$$

It should be emphasized that $u_m(\tau)$ for $m \geqslant 1$ is governed by the linear equation (8) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab. For the convergence of the above method we refer the reader to Liao's work [8] . If Eq. (1) admits unique solution, then this method will produce the unique solution. If equation (1) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

3 Applications

In order to assess the advantages and the accuracy of homotopy analysis method for solving linear and nonlinear Volterra integral equations,we will consider the following four examples.

Example 1 *Consider the integral equation*

Table 1: absolute error of HAM for different value of m and absolute error of DTM for different value of N

(x,t)	HAM(m=9)	DTM(N=10)	HAM(m=13)	DTM(N=14)
(0.2,0.2)	$0.69389e-17$	$0.10258e-15$	$0.69389e-17$	$0.10000e-20$
(0.2,0.8)	$0.27756e-16$	$0.42863e-09$	$0.27756e-16$	$0.53685e-14$
(0.4,0.6)	0	$0.36271e-10$	0	$0.14362e-15$
(0.4,1.0)	$0.11102e-15$	$0.99569e-08$	$0.11102e-15$	$0.30476e-12$
(0.6,0.2)	$0.13878e-16$	$0.30776e-15$	$0.13878e-16$	0
(0.6,0.8)	$0.11102e-15$	$0.12858e-08$	$0.16653e-15$	$0.16105e-13$
(0.8,0.4)	$0.55511e-16$	$0.83974e-12$	$0.55511e-16$	$0.65000e-18$
(0.8,0.8)	$0.13323e-14$	$0.17145e-08$	0	$0.21474e-13$
(1.0,0.6)	$0.35527e-15$	$0.90678e-10$	$0.11102e-15$	$0.35909e-15$
(1.0,1.0)	$0.28466e-12$	$0.24892e-07$	0	$0.76191e-12$

$$u(x,t)-\int_0^t\int_0^x (xy^2+\cos z)u(y,z)dydz = f(x,t), \quad x,t\in[0,1] \tag{13}$$

where $f(x,t) = x\sin t - \frac{1}{4}x^5 + \frac{1}{4}x^5\cos t - \frac{1}{4}x^2\sin^2 t$, which has the exact solution $u(x,t) = x\sin t$.[1]
To solve the Eq.(13) by means of homotopy analysis method, we choose the linear oprator

$$\mathcal{L}[\phi(x,t;p)] = \phi(x,t;p), \tag{14}$$

And we define a nonlinear operator as

$$\mathcal{N}[\phi(x,t;p)] = \phi(x,t;p) - \int_0^t\int_0^x (xy^2+\cos z)\phi(y,z;p)dydz - f(x,t), \tag{15}$$

Using above definitions (14) and (15), we obtain the mth-order deformation equations

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar\mathcal{H}(x,t)\mathcal{R}_m(\vec{u}_{m-1}), \quad (m\geq 1), \tag{16}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = u_{m-1}(x,t) - \int_0^t\int_0^x (xy^2+\cos z)u_{m-1}(y,z)dydz - f(x,t).$$

Now the solution of the mth-order deformation equations(16)

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar\mathcal{H}(x,t)\mathcal{R}_m(\vec{u}_{m-1}), (m\geq 1). \tag{17}$$

We start with an initial approximation $u_0(x,t) = 0$, by means of the above iteration formula (17) if $\mathcal{H}(x,t) = 1, \hbar = -1$, we can obtain directly the other components as

$$\begin{aligned} u_1(x,t) = \quad & \frac{1}{4}x(4\sin t - x^4 + x^4\cos t - x\sin^2 t), \\ u_2(x,t) = \quad & -\frac{1}{1440}x^2\left(45x^7t - 45x^7\sin t + 60x^4\sin t - 33x^4\sin 2t + 6x^4t + 360x^3\cos t\right. \\ & \left. +40x\sin t - 20x\sin 2t\cos t - 360\sin^2 t - 360x^3\right), \\ & \vdots \end{aligned}$$

In table (1) we have presented absolute error of HAM for different value of m and absolute error of DTM[1] for different value of N .

Example 2 *We consider the second example as*

$$u(x,t)-\int_0^t\int_0^x (xy+te^z)u(y,z)dydz = f(x,t), \quad x,t\in[0,1] \tag{18}$$

where $f(x,t) = xe^{-t} + t - \frac{1}{3}x^4 - xt + \frac{1}{3}x^4e^{-t} - \frac{1}{2}x^2t^2 - \frac{1}{4}x^3t^2 - xt^2e^t + xte^t$, *which has the exact solution* $u(x,t) = xe^{-t} + t$.*[1]*

Table 2: absolute error of HAM for different value of m and absolute error of DTM for different value of N

(x,t)	HAM(m=9)	DTM(N=10)	HAM(m=11)	DTM(N=12)
(0.2,0.6)	0	$0.402100e-13$	$0.22204e-15$	$0.693100e-16$
(0.2,1.0)	$0.22204e-15$	$0.299678e-10$	0	$0.143917e-12$
(0.4,0.4)	0	$0.419090e-15$	0	$0.330000e-18$
(0.4,0.8)	$0.22204e-15$	$0.333987e-11$	0	$0.102486e-13$
(0.6,0.8)	$0.66613e-15$	$0.500981e-11$	0	$0.153729e-13$
(0.6,1.0)	$0.13367e-12$	$0.899036e-10$	$0.44409e-15$	$0.431751e-12$
(0.8,0.4)	$0.44409e-15$	$0.838170e-15$	0	$0.640000e-18$
(0.8,0.8)	$0.17764e-13$	$0.667975e-11$	$0.44409e-15$	$0.204971e-13$
(1.0,0.6)	$0.15543e-14$	$0.201100e-12$	$0.44409e-15$	$0.346600e-15$
(1.0,1.0)	$0.35516e-10$	$0.149839e-9$	$0.37748e-14$	$0.719585e-12$

To solve the Eq.(18) by means of homotopy analysis method, we choose the linear oprator

$$\mathcal{L}[\phi(x,t;p)] = \phi(x,t;p), \tag{19}$$

And we define a nonlinear operator as

$$\mathcal{N}[\phi(x,t;p)] = \phi(x,t;p) - \int_0^t \int_0^x (xy + te^z)\phi(y,z;p)dydz - f(x,t), \tag{20}$$

Using above definitions (19) and (20), we obtain the mth-order deformation equations

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar\mathcal{H}(x,t)\mathcal{R}_m(\vec{u}_{m-1}), \quad (m \geq 1), \tag{21}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = u_{m-1}(x,t) - \int_0^t \int_0^x (xy + te^z)u_{m-1}(y,z)dydz - f(x,t).$$

Now the solution of the mth-order deformation equations(21)

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar\mathcal{H}(x,t)\mathcal{R}_m(\vec{u}_{m-1}), (m \geq 1). \tag{22}$$

We start with an initial approximation $u_0(x,t) = x$, by means of the above iteration formula (22) if $\mathcal{H}(x,t) = 1, \hbar = -1$, we can obtain directly the other components as

$$\begin{aligned}
u_1(x,t) = \quad & -\frac{1}{3} + \frac{1}{3}e^{-t} + \frac{1}{3}t)x^4 - \frac{1}{4}t^2x^3 + (\frac{1}{2}te^t - \frac{1}{2}t^2 - \frac{1}{2}t)x^2 \\
& +(-1 + e^{-t} - t + te^t + t^2e^t)x + t, \\
u_2(x,t) = \quad & \frac{1}{720}x\left(720t + 150x^2t - 150x^3t - 180xt^3e^{2t} + 360xt^2e^{2t} - 120x^2t^3e^t - 180xte^{2t}\right. \\
& +60x^2t^2e^{2t} - 30x^2te^{2t} + 360xt^2 + 40x^6 - 40x^6e^{-t} - 240x^3e^{-t} + 3x^4t^2 \\
& +720t^2e^t - 30x^4t^3 - 720te^t - 720x^3e^t + 90x^4 + 960x^3 + 96x^4t - 120x^2te^t \\
& +180xt - 360xt^2e^t - 120x^3t^2 - 40x^6t + 20x^6t^2 - 12x^5t^3 - 90x^4e^t + 48x^4t^2e^t \\
& \left.-6x^4te^t - 45x^3t^3e^t - 150x^3t^2e^t + 630x^3te^t + 120x^2t^2e^t + 180x^2t^2\right)
\end{aligned}$$

$$\vdots$$

In table (2) we have presented absolute error of HAM for different value of m and absolute error of DTM for different value of N .

Table 3: absolute error of HAM for different value of m and absolute error of DTM for different value of N

(x,t)	HAM(m=9)	DTM(N=10)	HAM(m=11)	DTM(N=12)
(0.1,0.7)	0	$0.525896e-11$	0	$0.163754e-13$
(0.2,0.3)	0	$0.182059e-14$	0	$0.104800e-17$
(0.3,0.9)	$0.42466e-14$	$0.764536e-09$	0	$0.392498e-11$
(0.4,1.0)	$0.55511e-16$	$0.437002e-08$	$0.58842e-14$	$0.276603e-10$
(0.5,0.8)	$0.44378e-10$	$0.576196e-09$	$0.34439e-12$	$0.234034e-11$
(0.6,1.0)	$0.49629e-08$	$0.983256e-08$	$0.89154e-10$	$0.622356e-10$
(0.7,0.6)	$0.10953e-07$	$0.468694e-10$	$0.22855e-09$	$0.107361e-12$
(0.8,1.0)	$0.24660e-05$	$0.174801e-07$	$0.13556e-06$	$0.110641e-09$
(0.9,0.5)	$0.91045e-06$	$0.103376e-10$	$0.42015e-07$	$0.164652e-13$
(1.0,1.0)	$0.36733e-03$	$0.273127e-07$	$0.49873e-04$	$0.172877e-09$

Example 3 *Consider the nonlinear Volterra integral equation*

$$u(x,t)-\int_0^t\int_0^x (y^2+e^{-2z})u^2(y,z)dydz = f(x,t), \quad x,t\in[0,1] \tag{23}$$

where $f(x,t)=x^2e^t+\frac{1}{14}x^7-\frac{1}{14}x^7e^{2t}-\frac{1}{5}x^5t$*, which has the exact solution* $u(x,t)=x^2e^t$*.[1]*

To solve the Eq.(23) by means of homotopy analysis method, we choose the linear oprator

$$\mathcal{L}[\phi(x,t;p)]=\phi(x,t;p), \tag{24}$$

And we define a nonlinear operator as

$$\mathcal{N}[\phi(x,t;p)]=\phi(x,t;p)-\int_0^t\int_0^x (y^2+e^{-2z})\phi^2(y,z;p)dydz - f(x,t), \tag{25}$$

Using above definitions (24) and (25), we obtain the mth-order deformation equations

$$\mathcal{L}[u_m(x,t)-\chi_m u_{m-1}(x,t)]=\hbar\mathcal{H}(x,t)\mathcal{R}_m(\vec{u}_{m-1}), \quad (m\geq 1), \tag{26}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1})=u_{m-1}(x,t)-\int_0^t\int_0^x (xy+te^z)\sum_{i=0}^{m-1}u_i(y,z)u_{m-1-i}(y,z)dydz - f(x,t).$$

Now the solution of the mth-order deformation equations(21)

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+\hbar\mathcal{H}(x,t)\mathcal{R}_m(\vec{u}_{m-1}),(m\geq 1). \tag{27}$$

We start with an initial approximation $u_0(x,t)=0$, by means of the above iteration formula (27) if $\mathcal{H}(x,t)=1,\hbar=-1$, we can obtain directly the other components as

$$\begin{aligned} u_1(x,t) = & \left(\frac{1}{14}-\frac{1}{14}e^{2t}\right)x^7-\frac{1}{5}tx^5+e^tx^2 \\ u_2(x,t) = & \ 0 \\ & \vdots \end{aligned}$$

In table (3) we have presented absolute error of HAM for different value of m and absolute error of DTM for different value of N .

Table 4: absolute error of HAM and DTM

(x,t)	HAM(m=3)	DTM(k=h=3)
(0.1,0.1)	$0.10043e-04$	$0.48040e-03$
(0.1,0.01)	$0.79591e-09$	$0.24888e-04$
(0.01,0.01)	$0.56859e-11$	$0.41570e-07$
(0.01,0.001)	$0.66613e-15$	$0.19972e-08$
(0.001,0.001)	$0.11102e-14$	$0.32991e-09$

Example 4 *Consider the nonlinear two dimensional integral equation*

$$u(x,y) = g(x,y) + 4\int_0^x \int_0^y (e^{x+y})u^2(t,s)dsdt, \quad x,y \in [0,1] \tag{28}$$

where $f(x,y) = e^{x+3y} + e^{3x+y} - e^{3(x+y)}$*, which has the exact solution* $u(x,y) = e^{x+y}$*.[6]*

To solve the Eq.(28) by means of homotopy analysis method, we choose the linear oprator

$$\mathcal{L}[\phi(x,y;p)] = \phi(x,y;p), \tag{29}$$

And we define a nonlinear operator as

$$\mathcal{N}[\phi(x,y;p)] = \phi(x,y;p) - g(x,y) - 4\int_0^x \int_0^y (e^{x+y})\phi^2(t,s;p)dsdt, \tag{30}$$

Using above definitions (29) and (30), we obtain the mth-order deformation equations

$$\mathcal{L}[u_m(x,y) - \chi_m u_{m-1}(x,y)] = \hbar\mathcal{H}(x,y)\mathcal{R}_m(\vec{u}_{m-1}), \quad (m \geq 1), \tag{31}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = u_{m-1}(x,y) - g(x,y) - 4\int_0^x \int_0^y (e^{x+y}) \sum_{i=0}^{m-1} u_i(t,s)u_{m-1-i}(t,s)dsdt.$$

Now the solution of the mth-order deformation equations(31)

$$u_m(x,y) = \chi_m u_{m-1}(x,y) + \hbar\mathcal{H}(x,y)\mathcal{R}_m(\vec{u}_{m-1}), (m \geq 1). \tag{32}$$

We start with an initial approximation $u_0(x,t) = e^x$, by means of the above iteration formula (32) if $\mathcal{H}(x,y) = 1, \hbar = -1$, we can obtain directly the other components as

$$\begin{aligned} u_1(x,t) = & \quad -e^x + e^{x+3y} + e^{3x+y} - e^{3(x+y)} + 2ye^{3x+y} - 2ye^{x+y} \\ u_2(x,t) = & \quad -4ye^{3x+y} + \frac{4}{3}e^{3x+4y} - 2e^{5x+2y} - \frac{2}{3}e^{5x+4y} + 4ye^{5x+2y} \\ & -8ye^{3x+2y} + 8e^{3x+2y} - \frac{28}{3}e^{3x+y} + \frac{8}{3}e^{5x+y} + 4ye^{x+y} \\ & -\frac{2}{3}e^{x+4y} - 6e^{x+2y} + 4ye^{x+2y} + \frac{20}{3}e^{x+y} \end{aligned}$$

$$\vdots$$

In table (4) we have presented absolute error of HAM and DTM .

4 Conclusion

In this paper, we have successfully developed HAM for solving a class of two-dimensional linear and nonlinear Volterra integral equations. It is apparently seen that HAM is a very powerful and efficient technique in finding approximate solutions for wide classes of linear and nonlinear problems. It is worth pointing out that this method presents a rapid convergence for the solutions. In conclusion, HAM provides accurate numerical solution for linear and nonlinear problems in comparison with the differential transform method. They also do not require large computer memory and discretization of the variables t and x.
Matlab has been used for computations in this paper.

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