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Pinning Synchronization of a Class of Complex Dynamical Network with Doupling Delay

Bin Zhuo *
Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu, 212013, China
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Abstract: On the basis of Lyapunov stability theory, local asymptotic stability control of a class of complex dynamical network with coupling delay is achieved by pinning controllers. Taking the Kronecker product multiplication as the main tool, through rigorous mathematical derivation, the asymptotic stability conditions of such complex dynamical network with coupling delay are obtained by applying some calculated controllers. At the same time, the whole network can be pinned to its equilibrium via placing pinning control on part of nodes of a specific complex network, numerical simulations are given to validate the accuracy of the theoretical analysis.

Keywords: complex network; synchronization; time-delay; Kronecker product; linear matrix inequalities

1 Introduction

As a brand of nonlinear science, complex dynamic network is a hot issue in recent years. The obtained results are becoming more and more important for engineering progress and further study of social phenomenon. In fact, many systems exist in science and technology and social phenomenon can be viewed as a complex network [1-4], such as Internet, World Wide Web, power grids, food webs, transportation networks, economic net, cellular networks, social networks, etc. In all research direction, the analysis and control of complex networks are well focus [5, 6], Wang and Chen introduced a uniform dynamical network model and investigated its synchronization and control [7, 8]. The common feature of the work in [9-12] is that there are no coupling delays in the network. However, in the real network because of the finite speed of transmission and spreading as well as congestions in signal transition, time-delay is a very inevitable and familiar phenomenon. Moreover, in a great many works [13,14], most authors use the method of adding controllers to all the nodes to make the complex networks get synchronization, but it is difficult to achieve in engineering practice. Based on current research, the paper researches complex dynamical network with coupling delay and investigates synchronization of a class of complex networks through applying simple and effective linear feedback controllers to part of network nodes using Kronecker product method.

The following is the paper's structure: In section 2, we give related definition and introduce the network model that we study. In section 3, we present the main research result that is controlled stabilities of the network with time delay. In section 4, examples of numerical simulations verify the accuracy of the theoretical analysis.

2 Model description and preliminaries

In this section, we introduce some mathematical preliminaries and a generalized complex delayed dynamical network model.

2.1 Mathematical preliminaries

Let R denote the field of real number. For a real matrix A , let A^T be the transpose of A , and A_{ij}denotes the i, jth entry of A. If A is a square matrix, $|A|$ is the determinant. A symmetric real matrix A is positive definite (semi-definite) if

*Corresponding author. *E-mail address*: zhuobin0501@126.com

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$x^TAx > 0(A \geq 0)$ for all nonzero x. We denote this as $A > 0(A \geq 0)$.We denote by I_n the n ×n real identity matrix. The kronecker product between two matrices is defined by

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & ... & A_{1n}B \\ A_{21}B & A_{22}B & ... & A_{2n}B \\ ... & ... & ... & ... \\ A_{m1}B & A_{m2}B & ... & A_{mn}B \end{pmatrix} \in R^{mp\times nq}$$

where $A \in R^{m\times n}$ and $B \in R^{p\times q}$.

In the following, we give several useful lemmas:

Lemma 1[15]: For any vectors $x, y \in R^n$ and positive definite matrix $Q \in R^{n\times n}$, the following matrix inequality holds $2x^Ty \leq x^TQx + y^TQ^{-1}y$.

Lemma 2[16]: Suppose $P \in R^{n\times n}$ is a symmetrical positive definite matrix, then

$$\lambda_{\min}(P)I_n \leq P \leq \lambda_{\max}(P)I_n.$$

Lemma 3[17]: Given $P = (P_{ij}) \in R^{n\times n}$ is symmetrical, and $P_{ii} < 0$ for all $i = 1, 2, ..., n$, if $P_{ii}+\sum\limits_{j=1,j\neq i}^{n} |P_{ij}| \leq -\varepsilon$, $i = 1, 2, ..., n$, where ε is a positive number, then $P \leq -\varepsilon I_n$.

2.2 Model description

In this paper, we consider a complex dynamical network consisting of Nidentical coupled nodes, with each node being an n-dimensional dynamical system:

$$\dot{x}_i(t) = Ax_i(t) + f(x_i(t)) + c\sum_{j=1}^{N} G_{ij}\Gamma x_j(t-\tau) \quad i = 1, 2, ..., N \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), ..., x_{in}(t))^T \in R^n$ are state variables of the ith delayed dynamical node, $Ax_i(t)$ is the linear part of the dynamical node with $A \in R^{n\times n}$, $f : R^n \to R^n$is the continuously differential nonlinear function, the constant $c > 0$ is the coupling strength, $\Gamma = (\gamma_{ij}) \in R^{n\times n}$ is a constant inner-coupling matrix for the nodes, $G = (G_{ij})_{N\times N} \in R^{N\times N}$ is a constant outer-coupling matrix of the nodes, in which G_{ij} is defined as follows: if there is a connection between the node i and the node $j(i \neq j)$,then $G_{ij} = G_{ji}$ =1; otherwise, $G_{ij} = G_{ji} = 0$, and the diagonal elements of matrix G are defined by $G_{ii} = -\sum\limits_{j=1,j\neq i}^{N} G_{ij} = -\sum\limits_{j=1,j\neq i}^{N} G_{ji}, i = 1, 2, ..., N$. It is assumed that network (1) is connected in the sense that there are no isolated clusters, that is, G is an irreducible matrix.

Using Kronecker product, network (1) can be rewritten as

$$\dot{x}(t) = Ax(t) + f(x(t)) + c(G \otimes \Gamma)x(t-\tau) \tag{2}$$

where $x(t) = (x_1(t), x_2(t), ..., x_N(t))^T \in R^{Nn}$, $\dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t), ..., \dot{x}_N(t))^T \in R^{Nn}$, $f(x(t)) = (f(x_1(t)), f(x_2(t)), ..., f(x_N(t))) \in R^{Nn} G \otimes \Gamma \in R^{Nn\times Nn}$denotes the Kronecker product of matrices G and Γ.

3 Synchronization of controlled dynamical network

Our task in the paper is to synchronize the states of the network (1) on the manifold defined (3) by introducing a delay-dependent linear controller into each individual node

$$x_1(t) = x_2(t) = ... = x_n(t) \to s(t) \qquad as \qquad t \to \infty \tag{3}$$

where $s(t) \in R^n$ is a solution of an isolated node, satisfy

$$\dot{s}(t) = As(t) + f(s(t)) \tag{4}$$

We assume that $s(t)$ is an arbitrary desired state which can be an equilibrium point, a nontrivial periodic orbit, or even a chaotic orbit. If the dynamical network (1) realizes synchronization, the synchronization state $S(t) = (s(t), ..., s(t))^T =$

$(1,...,1)^T \otimes s(t)$ is asymptotically stable; on the other hand, if the synchronization state $S(t)$ is asymptotically stable, then the dynamical network (1) realizes synchronization. The control objective is to make the states of the network (1) globally exponentially synchronize to $S(t)$.

In order to achieve the goal (3), feedback pinning controllers are applied onto a small portion $\delta(0 < \delta < 1)$ of nodes in network (1). Without loss of generality, let the first l nodes be selected to be pinned, where l is the integer part of the real number δN.

Thus, the controlled network can be described as

$$\dot{x}_i(t) = Ax_i(t) + f(x_i(t)) + c\sum_{j=1}^{N} G_{ij}\Gamma x_j(t-\tau) + u_i \quad i = 1, 2, ..., N \tag{5}$$

with the negative feedback controllers given by

$$u_i = -cd_i\Gamma(x_i(t-\tau) - s(t-\tau)) \quad i = 1, 2, ..., N \tag{6}$$

where $d_i > 0$ for $i = 1, 2, ..., l$ and $d_i = 0$ for $i = l+1, ..., N$. Substituting (6) into (5), one has

$$\dot{x}_i(t) = Ax_i(t) + f(x_i(t)) + c\sum_{j=1}^{N} G_{ij}\Gamma x_j(t-\tau) - cd_i\Gamma(x_i(t-\tau) - s(t-\tau)) \quad i = 1, 2, ..., N \tag{7}$$

Similar to (2), by using the Kronecker product, the controlled network (7) can be rewritten as

$$\dot{x}(t) = Ax(t) + f(x(t)) + c(G \otimes \Gamma)x(t-\tau) - c(D \otimes \Gamma)(x(t-\tau) - S(t-\tau))$$

where $D = diag(d_1, d_2, ..., d_N)$.

Let the errors be

$$e_i(t) = x_i(t) - s(t) \quad i = 1, 2, ..., N \tag{8}$$

where $e_i(t) \in R^n$ are the error variables between $x_i(t)$ and $s(t)$. Substituting (8) and (4) into (7), we obtain

$$\dot{e}_i(t) = Ae_i(t) + J(t)e_i(t) + c\sum_{j=1}^{N} G_{ij}\Gamma e_j(t-\tau) - cd_i\Gamma e_i(t-\tau) \quad i = 1, 2, ..., N \tag{9}$$

where $J(t)$ is the Jacobian of $f(x(t))$ at $s(t)$.

Using Kronecker product, we can rewrite (9) as

$$\dot{e}(t) = (I_N \otimes (A + J(t)))e(t) + c((G - D) \otimes \Gamma)e(t-\tau) \tag{10}$$

where $e(t) = (e_1(t), ..., e_N(t))^T \in R^{Nn}$ is the error variable between $x(t)$ and $S(t)$,

$$e(t-\tau) = (e_1(t-\tau), ..., e_N(t-\tau))^T \in R^{Nn}.$$

Theorem 1 *The synchronous state $S(t) = (s(t), ..., s(t))^T$ of the controlled dynamical network (7) is globally exponentially stable if there are two symmetrical positive definite matrix $P, Q \in R^{Nn}$ such that*

$$P + (I_N \otimes (A + J(t)))^T P + P(I_N \otimes (A + J(t))) + Qe^{\tau} + c^2 P((G - D) \otimes \Gamma)Q^{-1}((G - D) \otimes \Gamma)^T P \leq -\varepsilon I_{Nn}, \tag{11}$$

for any $\varepsilon > 0$. Consequently, the complex dynamical network (1) will globally exponentially realize synchronization with controllers.

Proof. Define a Lyapunov function as

$$V(t) = e^t e^T(t)Pe(t) + \int_{t-\tau}^{t} e^T(t)Qe(t)e^{\alpha+\tau} d\alpha$$

From Lemma 1 and 2, the derivative of $V(t)$ along the trajectory of (10) is

$$\begin{aligned}\dot{V}(t) &= e^t e^T(t)Pe(t) + e^t \dot{e}^T(t)Pe(t) + e^t e^T(t)P\dot{e}(t) + e^T(t)Qe(t)e^{t+\tau} - e^T(t-\tau)Qe(t-\tau)e^t \\ &= e^t e^T(t)Pe(t) + e^t((I_N \otimes (A+J(t)))e(t) + c((G-D)\otimes\Gamma)e(t-\tau))^T Pe(t) \\ &+ e^t e^T(t)P((I_N \otimes (A+J(t)))e(t) + c((G-D)\otimes\Gamma)e(t-\tau)) + e^T(t)Qe(t)e^{t+\tau} - e^T(t-\tau)Qe(t-\tau)e^t \\ &= e^t(e^T(t)Pe(t) + e^T(t)(I_N\otimes(A+J(t)))^T Pe(t) + e^T(t)P(I_N\otimes(A+J(t)))e(t) \\ &+ e^T(t)Qe^\tau e(t) + ce^T(t)P((G-D)\otimes\Gamma)e(t-\tau) + ce^T(t-\tau)((G-D)\otimes\Gamma)^T Pe(t) - e^T(t-\tau)Qe(t-\tau)) \\ &\le e^t e^T(t)(P + (I_N\otimes(A+J(t)))^T P + P(I_N\otimes(A+J(t))) + Qe^\tau \\ &+ c^2 P((G-D)\otimes\Gamma)Q^{-1}((G-D)\otimes\Gamma)^T P)e(t) \\ &\le -\varepsilon e^t \,||e(t)||^2 \\ &\le \frac{-\varepsilon}{\lambda_{\max}(P) + \tau e^\tau \lambda_{\max}(Q)} V(t)\end{aligned}$$

From the lyapunov stability theory, the error dynamical system (10) is globally exponentially stable about its zero solution. Consequently, the synchronization state $S(t)$ of network (7) is globally exponentially stable. The proof is thus completed. ■

4 Numerical and simulations

In this section, we will use an example to show the effectiveness of the main results above derived in this letter. We consider a 5-nodes network, in which each node is the third-order smooth Chua's circuits described in Refs. [18]

$$\dot{x}_{i1} = -k\alpha x_{i1} + k\alpha x_{i2} - k\alpha(ax_{i1}^3 + bx_{i1})$$

$$\dot{x}_{i2} = kx_{i1} - kx_{i2} + kx_{i3}$$

$$\dot{x}_{i3} = -k\beta x_{i2} - k\gamma x_{i3}$$

which is asymptotically stable at $s(t) = 0$, where $k = 1$, $\alpha = -0.1$, $\beta = -1$, $\gamma = 1$, $a = 1$,

$$b = 25.$$

The whole dynamical network is described by

$$\begin{pmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\\ \dot{x}_{i3}\end{pmatrix} = \begin{pmatrix}-24.9 & -0.1 & 0\\ 1 & -1 & 1\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}x_{i1}\\ x_{i2}\\ x_{i3}\end{pmatrix} + \begin{pmatrix}0.1x_{i1}^3\\ 0\\ 0\end{pmatrix} + c\sum_{j=1}^{5} G_{ij}\Gamma x_j(t-\tau), \quad i = 1, ..., 5 \tag{12}$$

That is $A = \begin{pmatrix}-24.9 & -0.1 & 0\\ 1 & -1 & 1\\ 0 & 1 & -1\end{pmatrix}$, $f(x_i(t)) = (0.1x_{i1}^3, 0, 0)^T$, $J(t) = 0$. The inner-coupling matrix is $\Gamma = diag\{1, 1, 1\}$, and the outer-coupling matrix is given by the following irreducible symmetric matrix:

$$G = \begin{pmatrix}-1 & 0 & 0 & 1 & 0\\ 0 & -2 & 0 & 1 & 1\\ 0 & 0 & -1 & 0 & 1\\ 1 & 1 & 0 & -2 & 0\\ 0 & 1 & 1 & 0 & -2\end{pmatrix}.$$

We want to stabilize this network onto the equilibrium point $s(t) = 0$ by applying the linear feedback pinning control (6) to a small fraction $\delta(0 < \delta < 1)$ of nodes. In the following, only the first node is selected to control for simplicity.

When the coupling strength $c = 2$, the feed back gain $d_1 = 3$ and $\tau = 0.15$, by using the Matlab LMI Toolbox, we found there exist two positive-definite matrices P, Q such that (11) holds, guaranteeing the globally exponentially stable of controlled network (12). The control results are shown in Fig. 1 when the delay is $\tau = 0.15$.

5 Conclusions

On the basis of Lyapunov stability theory, pinning synchronization of a class of complex dynamical network with coupling delay is achieved. The asymptotic stability conditions of such complex dynamical network with coupling delay are obtained by applying some calculated controllers. Finally, numerical simulations are given to validate the accuracy of the theoretical analysis.

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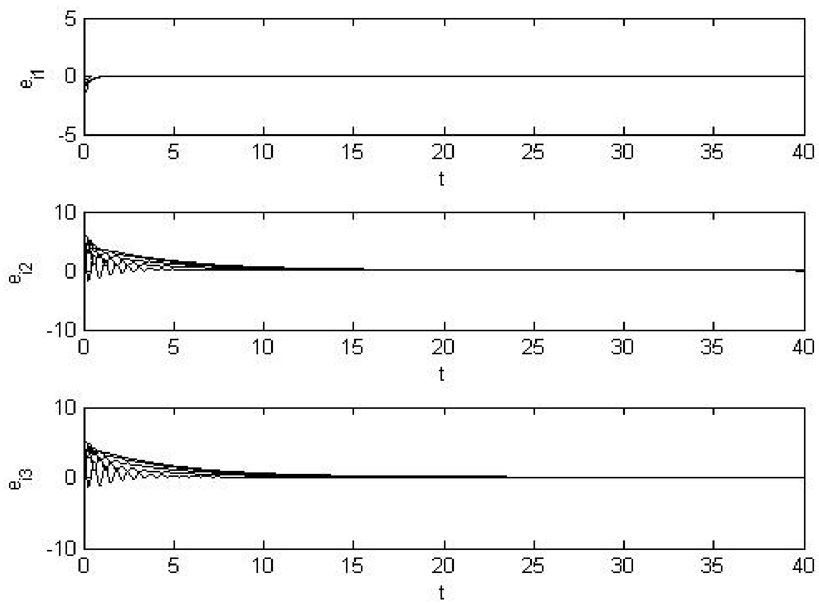

Figure 1: Synchronization errors of the delayed dynamical network with $\tau = 0.15$, where $e_{ij}(t) = x_{ij}(t) - s_j(t)$, $x_{ij}(t)$ is the jth state variable of the ith cell, and $s(t) = (s_1(t), s_2(t), s_3(t))^T$ is the synchronization solution with $1 \le i \le 5, 1 \le j \le 3$.

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