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Some New Variants of Chebyshev-Halley Methods Free from Second Derivative

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Abstract: In this paper, we present some new variants of Chebyshev-Halley methods free from second derivative for solving nonlinear equation of the type $f(x) = 0$, and show that the convergence orders of the proposed methods are three or four. Several numerical examples are given to illustrate the efficiency and performance of the new methods.

Keywords: nonlinear equations; Newton's method; variants of Chebyshev-Halley methods; order of convergence; efficiency index

1 Introduction

In this paper, we consider iterative methods to find a simple root α of nonlinear equations $f(x) = 0$, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

Newton's method is the well-known iterative method for finding α by using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

that converges quadratically in some neighborhood of α [1].

The classical Chebyshev-Halley methods [2] which improve Newton's method are given by

$$x_{n+1} = x_n - (1 + \frac{1}{2}\frac{Lf(x_n)}{1 - \beta Lf(x_n)})\frac{f(x_n)}{f'(x_n)} \tag{2}$$

where $Lf(x_n) = \frac{f(x_n)f''(x_n)}{f'(x_n)^2}$.

This family is known to be third-order, and includes, as particular cases, the classical Chebyshev's method ($\beta = 0$), Halley's method ($\beta = \frac{1}{2}$) and the super-Halley method ($\beta = 1$)(see [2-9,11] for more details). But the methods depend on second derivative $f''(x_n)$, which may be itself a serious and difficult problem in some cases. To overcome these drawbacks, we should develop iterative methods which are free from second derivative and whose orders are at least three, this is main motivation of this paper.

The paper is organized as follows. In section 2, we present some variants of Chebyshev-Halley methods free from second derivative which are obtained based on a two-step method. In section 3, we prove that the new methods have fourth-order or third-order. And in section 4, several numerical examples are given to illustrate the efficiency of the propose methods.

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2 Derivation of methods

Let us consider the sufficiently differentiable function $f : I \longrightarrow R$ for an open interval I. If $x_n \in I$, as [6], using the Taylor series, we have

$$f(y_n) \approx f(x_n) + (y_n - x_n)f'(x_n) + \frac{(y_n - x_n)^2}{2} f''(x_n) = \frac{1}{2}[\frac{f(x_n)}{f'(x_n)}]^2 f''(x_n), \quad (3)$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (4)$$

Therefore,

$$f''(x_n) \approx \frac{2f(y_n)f'(x_n)^2}{f(x_n)^2}. \quad (5)$$

Replacing $f''(y_n)$ by (5) in classical Chebyshev-Halley methods (2), we obtain a second-derivative-free variant of Chebyshev-Halley method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - (1 + \frac{f(y_n)}{f(x_n) - 2\beta f(y_n)})\frac{f(x_n)}{f'(x_n)} \end{cases}. \quad (6)$$

Let $\beta = 1$, we get a new variant of super-Halley method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - (1 + \frac{f(y_n)}{f(x_n) - 2f(y_n)})\frac{f(x_n)}{f'(x_n)} \end{cases}. \quad (7)$$

We also could use difference quotient to replace $f''(x_n)$, that is,

$$f''(x_n) \approx \frac{f'(x_n) - f'(y_n)}{x_n - y_n} = \frac{f'(x_n)^2 - f'(x_n)f'(y_n)}{f(x_n)}, \quad (8)$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$.

Replacing $f''(y_n)$ by (8) in classical Chebyshev-Halley methods (2), we obtain another second-derivative-free variant of Chebyshev-Halley method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - (1 + \frac{1}{2}\frac{f'(x_n) - f'(y_n)}{(1-\beta)f'(x_n) + \beta f'(y_n)})\frac{f(x_n)}{f'(x_n)} \end{cases}. \quad (9)$$

Let $\beta = 1$, we get another new variant of super-Halley method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - 1 + \frac{f'(x_n) - f'(y_n)}{2f'(y_n)})\frac{f(x_n)}{f'(x_n)} \end{cases}. \quad (10)$$

Motivate by formations of the methods (6) and (9), we introduce more general formats with parameters:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - (1 + \frac{Af(x_n) + Bf(y_n)}{Cf(x_n) + Df(y_n)})\frac{f(x_n)}{f'(x_n)} \end{cases} \quad (11)$$

and

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - (1 + \frac{Af'(x_n) + Bf'(y_n)}{Cf'(x_n) + Df'(y_n)})\frac{f(x_n)}{f'(x_n)}. \end{cases} \quad (12)$$

where A, B, C, D are the parameters.

3 Convergence analysis

For the methods defined by (11) and (12), we have the following convergence results.

Theorem 1 *Let $\alpha \in I$ be a simple root of the nonlinear equation $f(x) = 0$, where the function $f : I \longrightarrow R$ be sufficiently differentiable for the open interval I. If x_0 is sufficiently close to α, and A, B, C, D satisfy the conditions*

$$A = 0,\ C = B,\ D = -2B, \tag{13}$$

then the iterative method defined (11) is fourth-order, and its error equation is given by

$$e_{n+1} = (c_2^3 - c_2c_3)e_n^4 + O(e_n^5), \tag{14}$$

where $e_n = x_n - \alpha$, $c_k = f^{(k)}(\alpha)/k!f'(\alpha)$.

Proof. Let $\alpha \in I$ be a simple root of nonlinear equation $f(x) = 0$, x_0 be sufficiently close to α, x_n be defined by the iterative method (11), $e_n = x_n - \alpha$ and $c_k = f^{(k)}(\alpha)/k!f'(\alpha)$. Using Taylor expansion around α and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)], \tag{15}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3O(e_n^4)]. \tag{16}$$

Then

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5). \tag{17}$$

Thus, for y_n given in (4), we have

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 4c_2^3 + 3c_4)e_n^4 + O(e_n^5). \tag{18}$$

By expanding $f(y_n)$ about α, we have

$$f(y_n) = f'(\alpha)[c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 4c_2^3 + 3c_4)e_n^4 + O(e_n^5)]. \tag{19}$$

From(15) and (19), we get

$$\begin{aligned}\tfrac{Af(x_n)+Bf(y_n)}{Cf(x_n)+Df(y_n)} &= \tfrac{A}{C} + (\tfrac{B}{C} - \tfrac{AD}{C^2})c_2e_n + [(\tfrac{2B}{C} - \tfrac{2AD}{C^2})c_3 - (\tfrac{3B}{C} - \tfrac{3AD}{C^2} + \tfrac{BD}{C^2} - \tfrac{AD^2}{C^3})c_2^2]e_n^2 \\ &+Xe_n^3 + O(e_n^4),\end{aligned} \tag{20}$$

where X is a real valued function of A, B, C, D and c_k. Set

$$K_1 = \frac{A}{C},\ K_2 = \frac{B}{C} - \frac{AD}{C^2},\ K_3 = \frac{2B}{C} - \frac{2AD}{C^2},\ K_4 = \frac{3B}{C} - \frac{3AD}{C^2} + \frac{BD}{C^2} - \frac{AD^2}{C^3},$$

we easily obtain

$$\begin{aligned}(1 + \tfrac{Af(x_n)+Bf(y_n)}{Cf(x_n)+Df(y_n)})\tfrac{f(x_n)}{f'(x_n)} &= [(1 + K_1) + (K_2)c_2e_n + (K_3c_3 - K_4c_2^2)e_n^2 + Xe_n^3 + O(e_n^4)]\tfrac{f(x_n)}{f'(x_n)} \\ &= (1 + K_1)e_n + (K_2 - K_1 - 1)c_2e_n^2 \\ &+[(K_3 - 2K_1 - 2)c_3 + (2 + 2K_1 - K_2 - K_4)c_2^2]e_n^3 \\ &+[(1 + K_1)(7c_2c_3 - 4c_2^3 - 3c_4) + 2K_2c_2(c_2^2 - c_3) - c_2(K_3c_3 - K_4c_2^2) + X]e_n^4 \\ &+O(e_n^5).\end{aligned} \tag{21}$$

From (11) and (21), we now get

$$\begin{aligned}x_{n+1} &= x_n - (1 + \tfrac{Af(x_n)+Bf(y_n)}{Cf(x_n)+Df(y_n)})\tfrac{f(x_n)}{f'(x_n)} \\ &= \alpha - K_1e_n - (K_2 - K_1 - 1)c_2e_n^2 \\ &-[(K_3 - 2K_1 - 2)c_3 + (2 + 2K_1 - K_2 - K_4)c_2^2]e_n^3 \\ &-[(1 + K_1)(7c_2c_3 - 4c_2^3 - 3c_4) + 2K_2c_2(c_2^2 - c_3) - c_2(K_3c_3 - K_4c_2^2) + X]e_n^4 \\ &+O(e_n^5).\end{aligned} \tag{22}$$

Since $e_{n+1} = x_{n+1} - \alpha$, we then obtain the error equation

$$e_{n+1} = -K_1 e_n - (K_2 - K_1 - 1)c_2 e_n^2 - [(K_3 - 2K_1 - 2)c_3 + (2 + 2K_1 - K_2 - K_4)c_2^2]e_n^3 -$$
$$[(1 + K_1)(7c_2c_3 - 4c_2^3 - 3c_4) + 2K_2c_2(c_2^2 - c_3) - c_2(K_3c_3 - K_4c_2^2) + X]e_n^4 + O(e_n^5).$$

From the conditions $A = 0$, $C = B$, $D = -2B$, we have $K_1 = 0$, $K_2 - K_1 - 1 = 0$, $K_3 - 2K_1 - 2 = 0$, $2 + 2K_1 - K_2 - K_4 = 0$. So, the error equation is $e_{n+1} = (c_2^3 - c_2c_3)e_n^4 + O(e_n^5)$, this means that the method defined by (11) has fourth-order convergence. ■

Remark 2 *We could find that the method (11) under the condition (13) is same as the method (7) in fact.*

Theorem 3 *Let $\alpha \in I$ be a simple root of the nonlinear equation $f(x) = 0$, where the function $f(x) : I \longrightarrow R$ be sufficiently differentiable for the open interval I. If x_0 is sufficiently close to α, x_n is defined by the iterative method(12), and A, B, C, D satisfy the condition*

$$B = -A, C + D = 2A, \tag{23}$$

then the iterative method defined (12) is third-order, and its error equation is given by

$$e_{n+1} = (\frac{1}{2}c_3 + \frac{C}{A}c_2^2)e_n^3 + O(e_n^4), \tag{24}$$

where $e_n = x_n - \alpha$, $c_k = f^{(k)}(\alpha)/k!f'(\alpha)$.

Proof. Now expanding $f'(y)$ about α and using (18), we get

$$f'(y) = f'(\alpha)[1 + 2c_2^2e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + O(e_n^4)]. \tag{25}$$

From (16) and (25), we get

$$\begin{aligned} \frac{Af'(x_n)+Bf'(y_n)}{Cf'(x_n)+Df'(y_n)} &= \frac{A+B}{C+D} + [\frac{2A}{C+D} - \frac{2C(A+B)}{(C+D)^2}]c_2e_n \\ &+\{[\frac{3A}{C+D} - \frac{3C(A+B)}{(C+D)^2}]c_3 + [\frac{2B}{C+D} - \frac{2D(A+B)+4AC}{(C+D)^2} + \frac{4C^2(A+B)}{(C+D)^3}c_2^2]\}e_n^2 \\ &+O(e_n^3). \end{aligned} \tag{26}$$

Let

$$K_1 = \frac{A+B}{C+D},\ K_2 = \frac{2A}{C+D} - \frac{2C(A+B)}{(C+D)^2},$$
$$K_3 = \frac{3A}{C+D} - \frac{3C(A+B)}{(C+D)^2}, K_4 = \frac{2B}{C+D} - \frac{2D(A+B)+4AC}{(C+D)^2} + \frac{4C^2(A+B)}{(C+D)^3}.$$

We easily obtain

$$\begin{aligned} (1 + \frac{Af'(x_n)+Bf'(y_n)}{Cf'(x_n)+Df'(y_n)})\frac{f(x_n)}{f'(x_n)} &= [(1 + K_1) + K_2c_2e_n + (K_3c_3 + K_4c_2^2) + O(e_n^3)]\frac{f(x_n)}{f'(x_n)} \\ &= (1 + K_1)e_n + [K_2c_2 - c_2(1 + K_1)]e_n^2 \\ &+[K_3c_3 + (K_4 - K_2)c_2^2 + 2(1 + K_1)(c_2^2 - c_3)]e_n^3 + O(e_n^4). \end{aligned} \tag{27}$$

From (11) and (27), we have

$$\begin{aligned} x_{n+1} &= x_n - (1 + \frac{Af'(x_n)+Bf'(y_n)}{Cf'(x_n)+Df'(y_n)})\frac{f(x_n)}{f'(x_n)} \\ &= \alpha - K_1e_n - [K_2c_2 - c_2(1 + K_1)]e_n^2 - [K_3c_3 + (K_4 - K_2)c_2^2 + 2(1 + K_1)(c_2^2 - c_3)]e_n^3 + O(e_n^4). \end{aligned}$$

From the conditions $B = -A$, $C + D = 2A$, we have $K_1 = 0, K_2 - K_1 - 1 = 0$. So the error equation is $e_{n+1} = (\frac{1}{2}c_3 + \frac{C}{A}c_2^2)e_n^3 + O(e_n^4)$. This means that the method defined by (12) has third-order convergence. ■

Remark 4 *Set $A = -B = 1$, $C = 0$, $D = 2$, we get the variant of super-Halley method(10).*

Remark 5 *Set $A = -B = 1$, $C = 2$, $D = 0$, we get the variant of Chebyshev's method*

$$x_{n+1} = x_n - (1 + \frac{f'(x_n) - f'(y_n)}{2f'(x_n)})\frac{f(x_n)}{f'(x_n)}.$$

$f(x)$	IT NM	CM	SHM	VM1	VM2
$f_1, x_0 = 1.5$	4	3	3	2	3
$f_1, x_0 = 1$	5	4	3	3	3
$f_2, x_0 = 1$	4	3	3	2	3
$f_2, x_0 = -3$	6	4	3	3	4
$f_3, x_0 = -0.5$	11	/	6	4	5
$f_3, x_0 = -1$	6	5	4	4	4
$f_4, x_0 = 4$	7	5	5	4	5
$f_4, x_0 = 5$	8	6	4	4	4
$f_5, x_0 = 0.5$	6	4	4	3	4
$f_5, x_0 = -2$	13	/	/	5	6
$f_6, x_0 = 1$	6	5	3	3	3
$f_6, x_0 = 5$	7	5	5	4	4

Table 1: Numerical results

Remark 6 *Set $A = -B = 1$, $C = 1$, $D = 1$, we get the variant of Halley's method*

$$x_{n+1} = x_n - (1 + \frac{f'(x_n) - f'(y_n)}{f'(x_n) + f'(y_n)})\frac{f(x_n)}{f'(x_n)}.$$

Of course, we can obtain others methods under condition (23) by taking A, B, C and D as different values, respectively.

Remark 7 *It should be noted that the new methods require two functions and one first derivative evaluation or one function and two first derivative evaluations per iteration. If we consider the definition of efficiency index [10] as $p^{\frac{1}{m}}$, where p is the order of the method and m is the number of functional evaluations per iteration required by the method, we have that the new methods in this work have the efficiency index equal to $4^{\frac{1}{3}} \approx 1.587$ or $3^{\frac{1}{3}} \approx 1.442$, which is better than that of Newton's method.*

4 Numerical examples

In this section, we give the results of some numerical examples to demonstrate the efficiency of the new developed two-step iterative methods. All computations were done by using the Matrix-Lab package. We use the following stopping criteria for computer programs:

$(i) : |x_{n+1} - x_n| < \varepsilon$,

$(ii) : |f(x_{n-1})| < \varepsilon$.

For numerical illustrations we used the stopping criterion $\varepsilon = 10^{-15}$.

We compare the Newton's method(NM), the classical Chebyshev's method $(\beta = 0)$(CM), Super-Halley method $(\beta = 1)$(SHM); and the method(VM1)defined by (7), the method(VM2)defined by (10).

The examples are, $f_1(x) = x^3 + 4x^2 - 10$, $f_2(x) = x^2 - e^2 - 3x + 2$, $f_3(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$, $f_4(x) = (x-1)^3 - 2$, $f_5(x) = (x+2)e^x - 1$, $f_6(x) = \sin^2 x - x^2 + 1$.

As for the convergence criteria, it was required that the distance of two consecutive approximations δ for the zero was less than 10^{-15}. Also, the number of iterations to approximate the zero (IT) are displayed (see Table 1). In the table, the "/" means that the theorem is vain in this case.

In Table 1, as far as the numerical results are concerned, for most of the functions we tested, the presented methods can be competitive with Newton's method, Chebyshev's method and Super-Halley method. The methods obtained in this contribution is that they don't require the computation of second derivatives of the function.

5 Conclusion

In this work, we presented some variants of Chebyshev-Halley methods free from second derivative. In fact, we replace $Lf(x_n)$ in the Chebyshev-Halley method by using the combination of two functions $f(x_n)$ and $f(y_n)$ or two first derivatives $f'(x_n)$ and $f'(x_n)$. It will be interesting to consider other variants of Chebyshev-Halley methods obtained by replacing $Lf(x_n)$ with combinations of function and first derivative. Then, we may obtain other variants of Chebyshev-Halley methods with more accelerated convergence.

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References

[1] A. M. Ostrowski. Solution of Equations in Euclidean and Banach Space. *Academic Press, New York*(1973).

[2] J. M. Gutiérrez and M.A.Hernández. A family of Chebysheb-Halley type methods in Banach spaces. *Bull. Aust. Math.Soc.,* 55:(1997),113-130.

[3] J. M. Gutiérrez and M. A. Hernández. An acceleration of Newton's method:super-Halley method. *Appl. Math. Comput.,* 17:(2001),223-239.

[4] I. K. Argyros, D. Chen and Q. Qian. The Jarratt method in Banach space setting. *J.Comput. Appl.Math.,* 51:(1994),103-106.

[5] J. F. Traub. Iterative Methods for the Solution of Equations. *Chelsea Publishing Company, New York*(1997).

[6] M. Aslan Noor. Some iterative methods free from second derivatives for nonlinear equations. *Appl. Math. Comput.,* 192:(2007),101-106.

[7] J. S. Feng. A New Two-step Method for Solving Nonlinear Equations. *International Journal of Nonlinear Science.,* 8:(2009),40-44.

[8] Ali R. Soheili, S. A. Ahmadian and J. Naghipoor. A Family of Predictor-Corrector Methods Based on Weight Combination of Quadratures for Solving Nonlinear Equations. *International Journal of Nonlinear Science.,* 6:(2008),29-33.

[9] S. L. Wu, B. C. Shi and C. M. Huang. Relaxation Newton Iteration for A Class of Algebraic Nonlinear Systems. *International Journal of Nonlinear Science.,* 8:(2009),243-256.

[10] W. Gautschi. Numerical Analysis: An Introduction. *Birhäuser.,*(1997).

[11] E. Yusufoglu and A. Bekir. On the Extended Tanh Method Applications of Nonlinear Equations.*International Journal of Nonlinear Science.,* 4:(2007),10-16.