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On an Initial Boundary Value Problem for a Generalized Degasperis-Procesi Equation

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Abstract: In this paper, we study the following initial boundary value problem for a generalized Degasperis–Procesi equation

$$\begin{cases} u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = \gamma(u - u_{xx})_x, t \geq 0,\ x \in [0,\ 1] \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = 0, t \geq 0 \\ u(x,0) = u_0(x), x \in [0,\ 1] \end{cases}$$

We establish local well-posedness to it by using Kato's theorem for abstract quasilinear evolution equation of hyperbolic type. By the energy estimate, we present a blow up result.

Key words: Degasperis-Procesi equation; initial boundary value problem; blow up

1 Introduction

Recently, Degasperis and Procesi [1] derived a new nonlinear dispersive partial differential equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \tag{1}$$

which is called Degasperis-Procesi equation. After Eq.(1) was derived, many papers were devoted to its study. For example, Degasperis, Holm and Hone [2] showed that the Eq.(1) is integrable by deriving a Lax pair and a bi-Hamiltonian structure for it. Dullin et al. [3] showed that the Degasperis–Procesi equation can be regarded as a model for nonlinear shallow water dynamics. Yin proved local well-posedness to Eq.(1) with initial data $u_0 \in H^S(R)$, $S > 3$ on the line [4] and on the circle [5]. The global existence of strong solutions and weak solutions to Eq.(1) are investigated in [5]-[11]. For the blow-up of solution to Cauchy problem for Eq.(1) we refer to [8]-[11]. Vakhnenko and Parkes [12] obtained the traveling wave solutions to Eq.(1).The shock wave solutions to Eq.(1) are investigated in [13]-[15]. Lenells[16] classified all weak traveling wave solutions. For more exact solutions to Eq.(1), we refer to [17]-[21]. For the results of the generalized Degasperis-Procesi equations, we refer to [22, 23, 25]. Escher and Yin [24] studied the initial boundary value problem of Eq.(1) on the half line R^+ and on finite interval $[0, 1]$. On $[0, 1]$, they used the boundary contion $u(0, t) = u(1, t) = 0$, and obtained the local existence of solution to Eq.(1). They also obtained that the solution $u(x, t)$ to Eq.(1) with boundary condition $u(0, t) = u(1, t) = 0$ and initial conditions $u_0 \in H^S(0, 1) \cap H^1_0(0, 1)$ ($\frac{3}{2} < S \leq 3$) blows up in finite time if $u_0 \neq 0$.

In this paper, we are concerned with the following initial boundary value problem for a generalized form of Eq.(1)

$$\begin{cases} u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = \gamma(u - u_{xx})_x, \quad t \geq 0,\ x \in [0,\ 1] \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = 0, \quad t \geq 0 \\ u(x,0) = u_0(x), x \in [0,\ 1] \end{cases} \tag{2}$$

where γ is a real constant, γu_x denotes the dissipative term and γu_{xxx} denotes the dispersive effect. Obviously, when $\gamma = 0$, Eq.(2) is the well known Degasperis-Procesi Eq.(1).

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The remainder of the paper is organized as follows. In Section 2, we establish the local well-posedness to the closed-loop system (2) by Kato's theorem [26]. In Section 3, by using multiplier technique, we obtain the energy estimate. A blow-up phenomena is presented in this section.

We will use the following notation without further comment. $*$ for convolution; $L(Y, X)$ for all bounded linear operator from Banach space Y to X ($L(X)$ if $X = Y$); $\partial_x = \partial/\partial x$; $\Lambda = (1 - \partial_x^2)^{\frac{1}{2}}$; $H^S = H^S(0,1)$ with the norm $\|\cdot\|_{H^S} = \|\cdot\|_S$; $L^2 = L^2(0,1)$ with the norm $\|\cdot\|_{L^2}$; $H^S_{0,1} = \{u \in H^S(0,1) :\ u(0,t) = u(1,t)\}$; $[A,\ B]$ denotes the commutator of the linear operators A and B; $C^k(I;\ X)$for the space of all k times continuously differentiable functions defined on an interval I with values in Banach space X.

2 Local well-posedness

In this section , we will apply Kato's theorem [26] to esstablish the local well-posedness for the closed-loop system (2). For convenience, we state Kato's theorem in the form suitable for our purpose.

Consider the Cauchy problem associated to a quasilinear evolution equation

$$\begin{cases} \frac{du}{dt} + A(u)u = f(u) \in X, \quad t \geq 0 \\ u(0) = u_0 \in Y, \end{cases} \tag{1}$$

where $A(u)$ is a linear operator depending on the unknown u, and u_0 the initial value. To study the Cauchy problem (local in the time) associated to (1), we will make the following assumptions:

(X) X and $Y$are reflexive Banach spaces where $Y \subset X$, with the inclusion continuous and dense, and there is and isomorphism Q from Yonto X such that $\|\phi\|_Y = \|Q\phi\|_X$ for all $\phi \in Y$.

(A_1) Let Wbe an open ball centered in 0 and contained in Y. The linear operator $A(u)$belongs to $G(X, 1, \beta)$ where β is a real number, i.e., $-A(u)$generates a C_0- semigroup such that $\left\|e^{-sA(u)}\right\|_{B(X)} \leq e^{\beta s}$.

Note that if X is a Hilbert space, then $A \in G(X, 1, \beta)$ if and only if

(a) $(A\phi, \phi)_X \geq -\beta \|\phi\|_X^2$, $\forall \phi \in D(A)$,

(b) $(A + \lambda I)$ is onto for some (all) $\lambda > \beta$.

Under these conditions $A(u)$ is said to be quasi-m-accretive.

(A_2) The map $w \in W \to B(w) = [Q,\ A(w)]Q^{-1} \in L(X)$ is uniformly bounded and Lipschitz continuous, that is, there exist constants $\lambda_1, \mu_1 > 0$, such that for all $w,\ y \in W$,

$$\|B(w)\|_{L(X)} \leq \lambda_1,$$

(A_3) $X \subseteq D(A(w))$ for each $w \in W$ (so that $A(w)\mid_Y \in L(Y, X)$ by the Closed Graph theorem). Moreover, the map $w \in W \to A(w) \in L(Y, X)$ satisfies the following Lipschitz condition:

$$\|A(w) - A(y)\|_{L(Y,X)} \leq \mu_2 \|w - y\|_X ,$$

for all $w,\ y \in W$, where μ_2 is a non-negative constant.

(f) The function $f : W \to Y$is bounded, i.e., there is a constant $\lambda_2 > 0$ such that $\|f(w)\|_Y \leq \lambda_2$ for all $w \in W$, and the function $w \in X \to f(w)$is Lipschitz in X(resp. in Y), i.e.

$$\|f(w) - f(y)\|_X \leq \mu_3 \|w - y\|_X , \quad \forall w,\ y \in W,$$

$$\|f(w) - f(y)\|_Y \leq \mu_4 \|w - y\|_Y , \quad \forall w,\ y \in W,$$

where $\mu_3,\ \mu_4$ is non-negative constant.

We are now in position to state Kato's local well posedness result.

Theorem 1 (Kato's theorem [26]) *Assume conditions (X), (A_1)-(A_3) and (f) hold. Given $u_0 \in Y$, there is $T > 0$ and unique solution $u \in C([0,T]; Y) \cap C^1([0,T]; X)$ to (1) with $u(0) = u_0$. Moreover, the map $u_0 \in Y \to u \in C([0,T]; Y)$ is continuous.*

We now provide the framework in which we shall reformulate problem (2).

Let $m = u - u_{xx}$, then the first equation of (2) takes the form of a quasi-linear evolution equation of hyperbolic type

$$m_t + um_x + 3u_x m = \gamma m_x. \tag{2}$$

By using the operator $G(x) = \frac{\cosh(x-[x]-\frac{1}{2})}{2\sinh(\frac{1}{2})}$, where $[x]$ denotes the integer part of $x \in [0, 1]$, then $(1-\partial_x^2)^{-1}f = G * f, \forall f \in L^2$ and $G * m = u$. Using this identity, we can rewrite Eq.(2) as

$$u_t + uu_x + \partial_x(G * (\frac{3}{2}u^2)) = \gamma u_x$$

Then the closed-loop system (2) becomes

$$\begin{cases} u_t + uu_x + \partial_x(G * (\frac{3}{2}u^2)) = \gamma u_x, & t \geq 0,\ x \in [0,\ 1] \\ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = 0, & t \geq 0 \\ u(x,0) = u_0(x), & x \in [0,\ 1] \end{cases} \tag{3}$$

Theorem 2 *Given $u_0(x) \in H^2_{0,1}$, there exists a maximal time value $T = T(u_0(x)) > 0$ and a unique solution $u(x,t)$ to (3) such that $u = u(\cdot\ ;\ u_0) \in C([0,T); H^2_{0,1}) \cap C^1([0,T); H^1)$. Moreover, the solution depends continuously on the initial data, i.e., the mapping $u_0 \to u(\cdot\ ;\ u_0) : H^2_{0,1} \to C([0,T); H^2_{0,1}) \cap C^1([0,T); H^1)$ is continuous. Equivalently, system (2) has a unique solution $u(x,t) \in C([0,T); H^2_{0,1}) \cap C^1([0,T); H^1)$.*

Proof. Define $A(u)u = u\partial_x u - \gamma\partial_x u$, $f(u) = -\partial_x(G * (\frac{3}{2}u^2))$, $Q = \Lambda$, $X = H^1$, $Y = H^2_{0,1}$. Then the conditions (X), (A_1)-(A_3) and (f) in Theorem 2.1 are satisfied. Here we omit the details. One can see the similar argument in [26]. ■

3 blow up

By using multiplier technique, we obtain the following energy estimate of the closed-loop system (3).

Theorem 3 *Assume $u_0(x) \in H^2_{0,1}$. Let T be the maximal existence time of the solution $u(x,t)$ to (3) guaranteed by Theorem 2.2. Then we have $\|u(\cdot,t)\|^2_{L^2} \leq 4\|u_0\|^2_{L^2}$.*

Proof. For $u \in H^2_{0,1}$, we define the function v as follow

$$4v - v_{xx} = u.$$

Let $G_1(x) = e^{-2|x|}$ be the Green's function of the operator $4 - \partial_x^2$, then

$$u(x,t) = (G_1 * v)(x,t) = \int_0^1 e^{-2|x-y|}v(y,t)dy$$

Multiplying the first equation of (3) by $v - v_{xx}$, and integrating over $(0,1)$ by parts, we get

$$\begin{aligned} \int_0^1 u_t(v - v_{xx})dx = &- \int_0^1 uu_x(v - v_{xx})dx - \int_0^1 \partial_x(G * (\frac{3}{2}u^2))(v - v_{xx})dx \\ &+ \gamma\int_0^1 u_x(v - v_{xx})dx \end{aligned} \tag{1}$$

For the LHS of (1), by the periodic boundary conditions, using (1) and integration by parts, we have

$$\begin{aligned} \int_0^1 u_t(v - v_{xx})dx &= \int_0^1 (4v_t - v_{txx})(v - v_{xx})dx \\ &= -4v_tv_x\,\big|_0^1 - v_{tx}v\,\big|_0^1 + \int_0^1 (4v_tv + 5v_{tx}v_x + v_{txx}v_{xx})dx \\ &= \frac{1}{2}\frac{d}{dt}\int_0^1 (4v^2 + 5v_x^2 + v_{xx}^2)dx \end{aligned} \tag{2}$$

Similarly, for the RHS of (1), we obtain

$$\begin{aligned}
&-\int_0^1 uu_x(v-v_{xx})dx-\int_0^1 \partial_x(G*(\frac{3}{2}u^2))(v-v_{xx})dx+\gamma\int_0^1 u_x(v-v_{xx})dx\\
&=-\int_0^1 uu_x(v-v_{xx})dx-vG*(\frac{3}{2}u^2)\,|_0^1+\int_0^1 (G*(\frac{3}{2}u^2))v_xdx\\
&+\partial_x(G*(\frac{3}{2}u^2))v_x\,|_0^1-\int_0^1 \partial_x^2(G*(\frac{3}{2}u^2))v_xdx+\gamma\int_0^1 (4v_x-v_{xx})(v-v_{xx})dx\\
&=-\int_0^1 uu_x(v-v_{xx})dx+\int_0^1 \frac{3}{2}u^2v_xdx+2\gamma v^2|_0^1-2\gamma v_x^2|_0^1-\gamma vv_{xx}|_0^1+\frac{\gamma}{2}v_x^2|_0^1+\frac{\gamma}{2}v_{xx}^2|_0^1\\
&=-\int_0^1 uu_x(v-v_{xx})dx+\frac{3}{2}u^2v\,|_0^1-3\int_0^1 uu_xvdx\\
&=-\int_0^1 uu_x(4v-v_{xx})dx\\
&=-\int_0^1 u^2u_xdx\\
&=-\frac{1}{3}u^3\,|_0^1=0
\end{aligned}\tag{3}$$

It follows from (1)-(3) that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (4v^2+5v_x^2+v_{xx}^2)dx=0,$$

i.e.,

$$\begin{aligned}
&4\|v(\cdot,t)\|_{L^2}^2+5\|v_x(\cdot,t)\|_{L^2}^2+\|v_{xx}(\cdot,t)\|_{L^2}^2\\
&=4\|v(\cdot,0)\|_{L^2}^2+5\|v_x(\cdot,0)\|_{L^2}^2+\|v_{xx}(\cdot,0)\|_{L^2}^2
\end{aligned}\tag{4}$$

On the other hand, we have

$$\begin{aligned}
\|u(\cdot,t)\|_{L^2}^2&=\int_0^1 u^2dx=\int_0^1 (4v-v_{xx})^2dx\\
&=16\int_0^1 v^2dx-8vv_x\,|_0^1+8\int_0^1 v_x^2dx+\int_0^1 v_{xx}^2dx\\
&=16\|v(\cdot,t)\|_{L^2}^2+8\|v_x(\cdot,t)\|_{L^2}^2+\|v_{xx}(\cdot,t)\|_{L^2}^2\\
&\geq 4\|v(\cdot,t)\|_{L^2}^2+5\|v_x(\cdot,t)\|_{L^2}^2+\|v_{xx}(\cdot,t)\|_{L^2}^2.
\end{aligned}\tag{5}$$

Then it follows from (4) and (5) that

$$\begin{aligned}
\|u(\cdot,t)\|_{L^2}^2&=\int_0^1 u^2dx=16\|v(\cdot,t)\|_{L^2}^2+8\|v_x(\cdot,t)\|_{L^2}^2+\|v_{xx}(\cdot,t)\|_{L^2}^2\\
&\leq 4(4\|v(\cdot,t)\|_{L^2}^2+5\|v_x(\cdot,t)\|_{L^2}^2+\|v_{xx}(\cdot,t)\|_{L^2}^2)\\
&=4(4\|v(\cdot,0)\|_{L^2}^2+5\|v_x(\cdot,0)\|_{L^2}^2+\|v_{xx}(\cdot,0)\|_{L^2}^2)\\
&\leq 4\|u_0\|_{L^2}^2.
\end{aligned}\tag{6}$$

■

Now we present a blow-up result of solution to the closed-loop system (2) (or system (3)).

Theorem 4 *Assume $u_0(x)\in H_{0,1}^2$. Let T be the maximal existence time of the solution $u(x,t)$ to (3) guaranteed by Theorem 2.2. If there exists one point $x_0\in(0,1)$ such that $u_{xx}(x_0,t)=0$ and $u_0'(x_0)<-\sqrt{6}\|u_0\|_{L^2}$, then the corresponding solution to (3) blows up in finite time. Moreover, the maximal time of existence is estimated above by*

$$\frac{1}{\sqrt{6}\|u_0\|_{L^2}}\ln(\frac{h(0)-\sqrt{6}\|u_0\|_{L^2}}{h(0)+\sqrt{6}\|u_0\|_{L^2}}),$$

where $h(0)=u_0'(x_0)$.

Proof. Let $T > 0$ be the maximal time of existence of the solution $u(x,t)$ to (3) with the initial data $u_0(x) \in H^2_{0,1}$. Differentiating the first equation of (3) with respect to x, in view of $\partial_x^2 G * f = G * f - f$, we have

$$u_{tx} = -u_x^2 - uu_{xx} + \gamma u_{xx} + \frac{3}{2}u^2 - G * (\frac{3}{2}u^2). \tag{7}$$

Let $x = x_0$ in (7) and set $h(t) = u_x(x_0,t)$. Noting that $G * (\frac{3}{2}u^2) \geq 0$, we obtain

$$h'(t) = -h^2(t) + \frac{3}{2}u^2(x_0,t). \tag{8}$$

In view of (6), we have

$$u^2(x_0,t) \leq \|u\|^2_{L^\infty} \leq \|u\|^2_{L^2} \leq 4\|u_0\|^2_{L^2}. \tag{9}$$

It follows from (8) and (9) that

$$h'(t) \leq -h^2(t) + 6\|u_0\|^2_{L^2}.$$

Note that if $h(0) \leq -\sqrt{6}\|u_0\|_{L^2}$, then $h(t) \leq \sqrt{6}\|u_0\|_{L^2}$, for all $t \in [0,T)$. Therefore, from the above inequality we obtain

$$\frac{h(0) + \sqrt{6}\|u_0\|_{L^2}}{h(0) - \sqrt{6}\|u_0\|_{L^2}} e^{\sqrt{6}\|u_0\|_{L^2}} - 1 \leq \frac{\sqrt{6}\|u_0\|_{L^2}}{h(t) - \sqrt{6}\|u_0\|_{L^2}} < 0.$$

Due to $0 < \frac{h(0)+\sqrt{6}\|u_0\|_{L^2}}{h(0)-\sqrt{6}\|u_0\|_{L^2}} < 1$, then exists

$$T_0 \leq \frac{1}{\sqrt{6}\|u_0\|_{L^2}} \ln(\frac{h(0) - \sqrt{6}\|u_0\|_{L^2}}{h(0) + \sqrt{6}\|u_0\|_{L^2}})$$

such that

$$\lim_{t \to T_0} h(t) = -\infty.$$

■

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