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The implementation of Variational Homotopy Perturbation Method for Fisher's equation

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Abstract: In this paper, we consider the variational homotopy perturbation method (VHPM) to obtain the exact and numerical solutions of Fisher's partial differential equation. Comparisons are made among the variational iteration method (VIM), the exact solutions and the proposed method. The results reveal that the proposed method is very effective and simple and can be applied for other nonlinear problems in mathematical.

Keywords: Variational Homotopy Perturbation Method (VHPM); Fisher's equation; Lagrange multiplier

1 Introduction

In this paper, first we consider Fisher's partial differential equation

$$u_t = u_{xx} + \alpha u(1-u), \tag{1}$$

with the initial conditions

$$u(x,0) = f(x), \tag{2}$$

where $u_t = \frac{\partial u}{\partial t}, u_{xx} = \frac{\partial^2 u}{\partial x^2}$. This equation is encountered in chemical kinetics and population dynamics which includes problems such as nonlinear evolution of a population in a one-dimensional habitat and neutron population in a nuclear reaction and branching. Moreover the same equation occurs in logistic population growth models, flame propagation, neurophysiology, autocatalytic chemical reactions and branching Brownian motion processes [1].

2 Variational Homotopy Perturbation method

To convey the basic idea of the variational homotopy perturbation method [2, 3], we consider the following general differential equation:

$$Lu + Nu = g(x), \tag{3}$$

where L is a linear operator, N is a nonlinear operator and $g(x)$ is an inhomogeneous term. According to the variational iteration method [4–13], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\left\{Lu_n + N\tilde{u}_n - g(\tau)\right\}\, d\tau, \tag{4}$$

where $\lambda(\tau)$ is a Lagrange multiplier [4–13], which can be identified optimally via the variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta\tilde{u}_n = 0$ and (4) is called a correct functional. Now, we apply the homotopy perturbation method;

$$\sum_{i=0}^{\infty} p^i u_i = u_0 + p\int_0^x \lambda(\tau)\left\{N\left(\sum_{i=0}^{\infty} p^i\tilde{u}_i\right)\right\} d\tau - \int_0^x \lambda(\tau)g(\tau)\, d\tau, \tag{5}$$

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which is the variational homotopy perturbation method and is formulated by the coupling of variational iteration method and Adomian's polynomials. The embedding parameter $p \in [0, 1]$ can be considered as an expanding parameter [14–19]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [14–19] to obtain

$$f = \sum_{i=0}^{\infty} p^i u_i = u_0 + p\, u_1 + p^2 u_2 + \cdots . \tag{6}$$

If $p \to 1$, then (6) becomes the approximate solution of the form

$$u = \lim_{p\to 1} f = u_0 + u_1 + u_2 + \cdots . \tag{7}$$

A comparison of like powers of p gives solutions of various orders.

3 VHPM for Fisher's equation

In order to solve Fisher's equation (1) with initial conditions

$$u(x, 0) = f(x),$$

by means of VHPM, we consider

$$L(u) = u_t, \tag{8}$$

$$N(u) = -u_{xx} - \alpha u\,(1-u), \tag{9}$$

where L is a linear and N is a nonlinear operator. According to the variational iteration method [4–13], we can construct a correct functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\,\{u_{n_\tau} - \tilde{u}_{n_{xx}} - \alpha \tilde{u}_n(1-\tilde{u}_n)\}\; d\tau, \tag{10}$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = -1$, which yields the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{u_{n_\tau} - u_{n_{xx}} - \alpha u_n(1-u_n)\}\; d\tau. \tag{11}$$

Applying the variational homotopy perturbation method, we have:

$$\begin{aligned} u_0 \;&+\; p\, u_1 + p^2 u_2 + \cdots = f(x) + p\int_0^t (u_{0_{xx}} + p\, u_{1_{xx}} + p^2 u_{2_{xx}} + \cdots)\, d\tau \\ &+\; \alpha p \int_0^t (u_0 + p\, u_1 + p^2 u_2 + \cdots)\, d\tau \; - \alpha p \int_0^t (u_0 + p\, u_1 + p^2 u_2 + \cdots)^2\, d\tau. \end{aligned} \tag{12}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned} p^0 : u_0(x,t) &= f(x), \\ p^1 : u_1(x,t) &= \int_0^t u_{0_{xx}}\, d\tau + \alpha \int_0^t u_0\, d\tau - \alpha \int_0^t {u_0}^2\, d\tau, \\ p^2 : u_2(x,t) &= \int_0^t u_{1_{xx}}\, d\tau + \alpha \int_0^t u_1\, d\tau - \alpha \int_0^t 2u_0 u_1\, d\tau, \\ p^3 : u_3(x,t) &= \int_0^t u_{2_{xx}}\, d\tau + \alpha \int_0^t u_2\, d\tau - \alpha \int_0^t (2u_0 u_2 + {u_1}^2)\, d\tau, \\ &\vdots \end{aligned}$$

So we obtain the components which constitute $u(x,t)$, thus we will have:

$$u(x,t) = u_0 + u_1 + u_2 + \cdots$$

For later numerical computation, we let the expression

$$\varphi_n = \sum_{i=0}^{n} u_i(x,t), \tag{13}$$

to denote the n-term approximation to $u(x,t)$.

4 Implementation of the method

In this section three important cases of Fisher's equation which correspond to some real physical processes will be investigated to show the reliability of the proposed scheme.

Example 1. We consider the equation (1) with $\alpha = 1$, $u(x,0) = \mu$, by using the equation (12) we have:

$$u_0 + p\,u_1 + p^2 u_2 + \cdots = \mu + p\int_0^t \left(u_{0_{xx}} + p\,u_{1_{xx}} + p^2 u_{2_{xx}} + \cdots\right)\,d\tau \tag{14}$$

$$+ p\int_0^t (u_0 + p\,u_1 + p^2 u_2 + \cdots)\,d\tau - p\int_0^t (u_0 + p\,u_1 + p^2 u_2 + \cdots)^2\,d\tau. \tag{15}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned}
p^0 : u_0(x,t) &= \mu, \\
p^1 : u_1(x,t) &= t\,\mu\,(1-\mu), \\
p^2 : u_2(x,t) &= \frac{t^2}{2!}\,\mu\,(1-\mu)(1-2\mu), \\
p^3 : u_3(x,t) &= \frac{t^3}{3!}\,\mu\,(1-\mu)\left(1-6\mu+6\mu^2\right), \\
p^4 : u_4(x,t) &= \frac{t^4}{4!}\,\mu\,(1-\mu)(1-2\mu)\left(1-12\mu+12\mu^2\right),
\end{aligned}$$

$\vdots$

So we obtain the components which constitute $u(x,t)$, thus we will have:

$$\begin{aligned}
u(x,t) &= u_0 + u_1 + u_2 + \cdots = \mu + t\,\mu\,(1-\mu) + \frac{t^2}{2!}\,\mu\,(1-\mu)(1-2\mu) \\
&+ \frac{t^3}{3!}\,\mu\,(1-\mu)\left(1-6\mu+6\mu^2\right) + \frac{t^4}{4!}\,\mu\,(1-\mu)(1-2\mu)\left(1-12\mu+12\mu^2\right) + \cdots,
\end{aligned}$$

so the exact solution is obtained as follows:

$$u(x,t) = \frac{\mu\,e^t}{1-\mu+\mu\,e^t}, \tag{16}$$

as presented in[20].

Example 2. We consider the equation (1) with

$$\alpha = 6, u(x,0) = \frac{1}{(1+e^x)^2},$$

by using the equation (12) we have:

$$u_0 + p\,u_1 + p^2 u_2 + \cdots = \frac{1}{(1+e^x)^2} + p\int_0^t \left(u_{0_{xx}} + p\,u_{1_{xx}} + p^2 u_{2_{xx}} + \cdots\right)\,d\tau \tag{17}$$

$$+ 6\,p\int_0^t \left(u_0 + p\,u_1 + p^2 u_2 + \cdots\right)\,d\tau - 6p\int_0^t \left(u_0 + p\,u_1 + p^2 u_2 + \cdots\right)^2\,d\tau. \tag{18}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned} p^0 : u_0(x,t) &= \frac{1}{(1+e^x)^2}, \\ p^1 : u_1(x,t) &= 10\, t\, \frac{e^x}{(1+e^x)^3}, \\ p^2 : u_2(x,t) &= 50\, \frac{t^2}{2!}\, \frac{e^x\,(2e^x-1)}{(1+e^x)^4}, \\ p^3 : u_3(x,t) &= 250\, \frac{t^3}{3!}\, \frac{e^x\left(4e^{2x}-7e^x+1\right)}{(1+e^x)^5}, \\ p^4 : u_4(x,t) &= 1250\, \frac{t^4}{4!}\, \frac{e^x\left(8e^{3x}-33e^{2x}+18e^x-1\right)}{(1+e^x)^6}, \\ &\vdots \end{aligned}$$

So we obtain the components which constitute $u(x,t)$, thus we will have:

$$\begin{aligned} u(x,t) &= u_0+u_1+u_2+\cdots = \frac{1}{(1+e^x)^2} + 10\, t\, \frac{e^x}{(1+e^x)^3} + 50\, \frac{t^2}{2!}\, \frac{e^x\,(2e^x-1)}{(1+e^x)^4} \\ &+ 250\, \frac{t^3}{3!}\, \frac{e^x\left(4e^{2x}-7e^x+1\right)}{(1+e^x)^5} + \cdots, \end{aligned}$$

so the exact solution is obtained as follows:

$$u(x,t) = \frac{1}{\left(1+e^{(x-5t)}\right)^2}, \tag{19}$$

as presented in [20].

Example 3. Now we consider the more general case of example (2) with

$$u(x,0) = \frac{1}{\left(1+e^{\sqrt{\frac{\alpha}{6}}x}\right)^2},$$

by using the equation (12) we have:

$$u_0 + p\,u_1 + p^2u_2 + \cdots = \frac{1}{\left(1+e^{\sqrt{\frac{\alpha}{6}}x}\right)^2} + p\int_0^t \left(u_{0_{xx}} + p\,u_{1_{xx}} + p^2u_{2_{xx}} + \cdots\right)\, d\tau \tag{20}$$

$$+ \alpha p \int_0^t \left(u_0 + p\,u_1 + p^2u_2 + \cdots\right)\, d\tau - \alpha p \int_0^t \left(u_0 + p\,u_1 + p^2u_2 + \cdots\right)^2\, d\tau. \tag{21}$$

Comparing the coefficient of like powers of p, we have:

$$\begin{aligned} p^0 : u_0(x,t) &= \frac{1}{\left(1+e^{\sqrt{\frac{\alpha}{6}}x}\right)^2}, \\ p^1 : u_1(x,t) &= t\, e^{\sqrt{\frac{\alpha}{6}}x}\, \frac{\frac{2}{3}\,\alpha\, e^{\sqrt{\frac{\alpha}{6}}x} - \frac{\alpha}{3} + 12 + 6e^{\sqrt{\frac{\alpha}{6}}x}}{\left(1+e^{\sqrt{\frac{\alpha}{6}}x}\right)^4}, \\ &\vdots \end{aligned}$$

So we obtain the components which constitute $u(x,t)$, thus we will have:

$$u(x,t) = u_0+u_1+u_2+\cdots = \frac{1}{\left(1+e^{\sqrt{\frac{\alpha}{6}}x}\right)^2} + t\, e^{\sqrt{\frac{\alpha}{6}}x}\, \frac{\frac{2}{3}\,\alpha\, e^{\sqrt{\frac{\alpha}{6}}x} - \frac{\alpha}{3} + 12 + 6e^{\sqrt{\frac{\alpha}{6}}x}}{\left(1+e^{\sqrt{\frac{\alpha}{6}}x}\right)^4} + \cdots,$$

so the exact solution is obtained as follows:

$$u(x,t) = \frac{1}{\left(1+e^{\left(\sqrt{\frac{\alpha}{6}}x-\frac{5}{6}\alpha t\right)}\right)^2}, \tag{22}$$

as presented in [20].

In what follows, we present the absolute errors between φ_{5VHPM} and the exact solution and the absolute errors between the 5-iterate of VIM ($u_{5_{VIM}}$) and the exact solution for the values of $t = 0.1$, $x = 0(0.1)0.5$ and $\alpha = 6$.

Table 1: The numerical results for $\varphi_{5_{VHPM}}$ and $u_{5_{VIM}}$ in comparison with the exact solution of u.

x	$\lvert u-\varphi_{5_{VHPM}}\rvert$	$\lvert u-u_{5_{VIM}}\rvert$
0.0	$2.0723e-005$	$3.9024e-007$
0.1	$2.3329e-005$	$4.0391e-007$
0.2	$2.4263e-005$	$1.0203e-006$
0.3	$2.3508e-005$	$1.3595e-006$
0.4	$2.1223e-005$	$1.3781e-006$
0.5	$1.7715e-005$	$1.0952e-006$

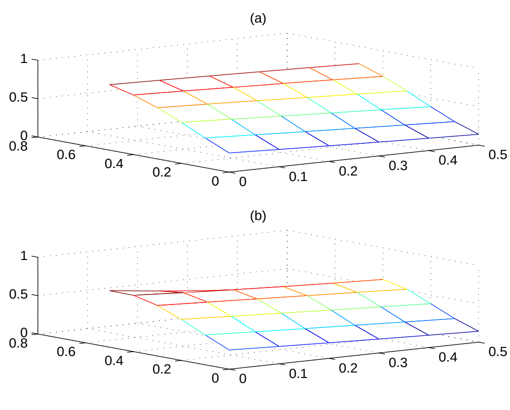

Figure 1: The exact solution (a) $u(x,t)$ and the approximate solution (b) $\varphi_{5_{VHPM}}$ with fixed value $\alpha = 6$ at $x = 0(0.1)0.5$ and $t = 0(0.1)0.5$.

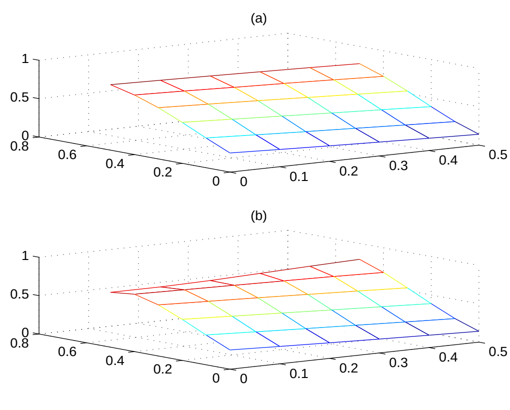

Figure 2: The exact solution (a) $u(x,t)$ and the 5-iterate of VIM (b) $u_{5_{VIM}}$ with fixed value $\alpha = 6$ at $x = 0(0.1)0.5$ and $t = 0(0.1)0.5$.

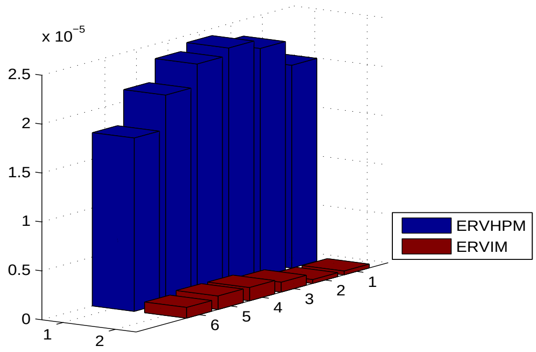

Figure 3: The absolute error between the exact solution and $\varphi_{5_{VHPM}}$ and the absolute error between the exact solution and $u_{5_{VIM}}$, with fixed value $\alpha = 6$ at $x = 0(0.1)0.5$ and $t = 0.1$.

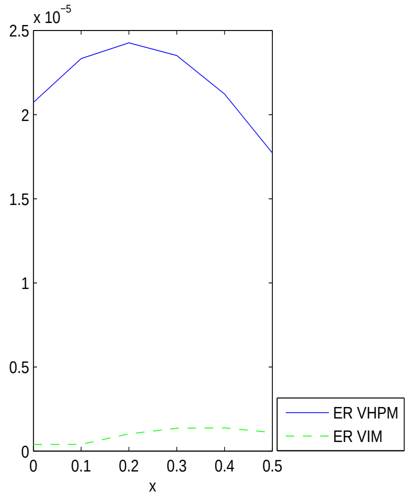

Figure 4: The absolute error between the exact solution and $\varphi_{5_{VHPM}}$ and the absolute error between the exact solution and $u_{5_{VIM}}$, with fixed value $\alpha = 6$ at $x = 0(0.1)0.5$ and $t = 0.1$.

The numerical results reveal that the VHPM is easy to implement and reduces the computational work to a tangible level while still maintaining a very higher level of accuracy.

5 Conclusion

In this paper, varational homotopy perturbation method is proposed for solving Fisher's equation. The small amount of computation compared to that required in other methods such as the variational iteration method and the rapid convergence, show that the method is reliable and provides a significant improvement in solving partial differential equations over existing methods. The computations in this paper are done by MATLAB software.

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