

## On a New Aftertreatment Technique for Differential Transformation Method and Its Application to Non-linear Oscillatory Systems

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**Abstract:** In this paper, a new aftertreatment (AT) technique is proposed to deal with the truncated series derived by the differential transformation method (DTM) to obtain approximate periodic solutions. The proposed aftertreatment technique splits into two types, named as (Sine-AT technique, SAT) and (Cosine-AT technique, CAT). The non-linear Duffing equation under two different initial conditions and Van der pol equation are chosen to illustrate the method of solution. The results obtained in this study reveal that the proposed aftertreatment technique is very effective and convenient for non-linear oscillatory systems. In comparison with the previous aftertreatment technique proposed by [18,19], the present technique can be used easily and in a straightforward manner without any need to Padé approximants or Laplace transform.

**Keywords:** Non-linear oscillatory systems; Duffing equation; periodic solutions; Adomian decomposition method; differential transformation method; aftertreatment technique

### 1 Introduction

Since the beginning of the 1980s, Adomian [1] has developed a so-called decomposition method. In 1986, Zhou [2] has also presented a so-called differential transformation method. These two methods have been used extensively during the last two decades to solve effectively and easily various linear and nonlinear ordinary and partial differential equations [3-20]. The main advantage of these methods is that they can be applied directly to differential equations without requiring linearization, discretization or perturbation. A well known fact is that, the Adomian decomposition method (ADM) and the differential transformation method (DTM) give the solution in the form of a truncated series. Unfortunately, in the case of oscillatory systems, the truncated series obtained by the two methods is periodic only in a very small region [18,20], but in a wider range it is not so. To overcome this difficulty, an aftertreatment (AT) technique has been proposed in [18,19] to obtain approximate periodic solutions in a wider range. Their technique is based on using Padé approximants, Laplace transform and its inverse to deal with the truncated series. Although their AT technique was found to be effective in many cases, it has some disadvantages, not only a huge amount of computational work is required to result accurate approximations for the periodic solutions but also there is a difficulty of obtaining the inverse Laplace transform which will greatly restrict the application area of their technique. In this research, we propose a new aftertreatment technique to deal with the truncated series obtained from series solution methods. This new aftertreatment technique avoids the use of Padé approximants and Laplace transform, it may be one of its advantages over the previous one. The rest of this work is organized as follows: in the next section, a brief description of DTM is provided; in Sections 3 and 4, the method developed in the present work, Cosine-AT and Sine-AT are discussed in detail; in Section 5, the proposed techniques are implemented to the approximate periodic solution of the Duffing equation; in Section 6, the techniques is applied to Van der Pol equation; in Section 7, some conclusions are given.

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## 2 One-dimensional differential transform

Differential transform of a function  $x(t)$  is defined as follows:

$$X(k) = \frac{1}{k!} \left[ \frac{d^k x(t)}{dt^k} \right]_{t=0}, \tag{1}$$

where  $x(t)$  and  $X(k)$  are the original and transformed functions, respectively. Differential inverse transform of  $X(k)$  is defined as

$$x(t) = \sum_{k=0}^{\infty} X(k)t^k. \tag{2}$$

So

$$x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[ \frac{d^k x(t)}{dt^k} \right]_{t=0}. \tag{3}$$

Eq. (3) implies that the concept of differential transform is derived from Taylor series expansion. In actual applications, the function  $x(t)$  is expressed by a truncated series and Eq. (2) can be written as

$$\Phi_N(t) = \sum_{k=0}^N X(k)t^k. \tag{4}$$

Some of the fundamental mathematical operations performed by differential transform method are listed in Table 1.

Table 1: The fundamental operations of one-dimensional DTM.

Original function $x(t)$	Transformed function $X(k)$
$\alpha u(t) \pm \beta v(t)$	$\alpha U(k) \pm \beta V(k)$
$\frac{d^m x(t)}{dt^m}$	$\frac{(k+m)!}{k!} X(k+m)$
$u(t)v(t)$	$\sum_{l=0}^k U(l)V(k-l)$
$u(t)v(t)w(t)$	$\sum_{l=0}^k \sum_{m=0}^{k-l} U(l)V(m)W(k-l-m)$
$u(t) \int_0^t v(t)dt$	$\sum_{l=1}^k U(k-l) \frac{V(l-1)}{l}, k \geq 1$
$t^m$	$\delta(k-m) = 1$ if $k=m, 0$ if $k \neq m$
$\exp(t)$	$\frac{1}{k!}$
$\sin(\lambda t + \omega)$	$\frac{\lambda^k}{k!} \sin\left(\frac{k\pi}{2} + \omega\right)$
$\cos(\lambda t + \omega)$	$\frac{\lambda^k}{k!} \cos\left(\frac{k\pi}{2} + \omega\right)$

## 3 Cosine-aftertreatment (CAT) technique

If the truncated series given by Eq. (4) is expressed in even-powers, only, of the independent variable  $t$ , i.e.,

$$\Phi_N(t) = \sum_{k=0}^N X(2k)t^{2k}, \quad X(2k+1) = 0, \forall k = 0, 1, \dots, \frac{N}{2} - 1, \text{ where } N \text{ is even,} \tag{5}$$

then the Cosine-aftertreatment technique (CAT-technique) is based on the assumption that this truncated series can be expressed as another finite series in terms of the cosine trigonometric functions with different amplitudes and arguments:

$$\Phi_N(t) = \sum_{j=1}^n \lambda_j \cos(\Omega_j t), \text{ where } n \text{ is finite.} \tag{6}$$

By expanding both sides of Eq. (6) as power series of  $t$  and equating the coefficients of like powers, we get

$$\begin{aligned} t^0 : & \sum_{j=1}^n \lambda_j = X(0), \\ t^2 : & \sum_{j=1}^n \lambda_j \Omega_j^2 = -2!X(2), \\ t^4 : & \sum_{j=1}^n \lambda_j \Omega_j^4 = 4!X(4), \\ t^6 : & \sum_{j=1}^n \lambda_j \Omega_j^6 = -6!X(6), \\ & \dots \end{aligned} \tag{7}$$

In practical applications, it is sufficient to express the truncated series  $\Phi_N(t)$  in terms of two or three cosines with different amplitudes and arguments. If we choose to express  $\Phi_N(t)$  as an approximate periodic solution in terms of two cosines with two different amplitudes,  $\lambda_1$ ,  $\lambda_2$  and two different arguments,  $\Omega_1$ ,  $\Omega_2$ , we can rewrite Eq. (6) at  $N = 6$ ,  $n = 2$ , as:

$$\Phi_6(t) = \sum_{j=1}^2 \lambda_j \cos(\Omega_j t). \quad (8)$$

In this case, the four unknowns  $\lambda_1$ ,  $\lambda_2$ ,  $\Omega_1$  and  $\Omega_2$  can be determined by solving the following system of non-linear algebraic equations, analytically:

$$\begin{aligned} \lambda_1 + \lambda_2 &= X(0), \\ \lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 &= -2!X(2), \\ \lambda_1 \Omega_1^4 + \lambda_2 \Omega_2^4 &= 4!X(4), \\ \lambda_1 \Omega_1^6 + \lambda_2 \Omega_2^6 &= -6!X(6). \end{aligned} \quad (9)$$

Moreover, If we choose to express  $\Phi_N(t)$  as more accurate periodic solution in terms of three cosines, we can rewrite Eq. (6) at  $N = 10$ ,  $n = 3$ , as:

$$\Phi_{10}(t) = \sum_{j=1}^3 \lambda_j \cos(\Omega_j t). \quad (10)$$

In this case, the six unknowns  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$ ,  $\Omega_1$ ,  $\Omega_2$  and  $\Omega_3$  can be determined by solving the following system of non-linear algebraic equations, numerically:

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= X(0), \\ \lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 + \lambda_3 \Omega_3^2 &= -2!X(2), \\ \lambda_1 \Omega_1^4 + \lambda_2 \Omega_2^4 + \lambda_3 \Omega_3^4 &= 4!X(4), \\ \lambda_1 \Omega_1^6 + \lambda_2 \Omega_2^6 + \lambda_3 \Omega_3^6 &= -6!X(6), \\ \lambda_1 \Omega_1^8 + \lambda_2 \Omega_2^8 + \lambda_3 \Omega_3^8 &= 8!X(8), \\ \lambda_1 \Omega_1^{10} + \lambda_2 \Omega_2^{10} + \lambda_3 \Omega_3^{10} &= -10!X(10). \end{aligned} \quad (11)$$

## 4 Sine-aftertreatment (SAT) technique

If the truncated series given by Eq. (4) is obtained as a finite polynomial in odd-powers of the independent variable  $t$ , i.e.,

$$\Phi_N(t) = \sum_{k=0}^N X(2k+1)t^{2k+1}, \quad X(2k) = 0, \quad \forall k = 0, 1, \dots, \frac{N-1}{2}, \quad \text{where } N \text{ is odd}, \quad (12)$$

then the Sine-aftertreatment technique (SAT-technique) is based on the assumption that this truncated series can be expressed as an approximate periodic solution, in terms of the sine trigonometric functions with different amplitudes and arguments:

$$\Phi_N(t) = \sum_{j=1}^n \mu_j \sin(\Psi_j t), \quad \text{where } n \text{ is finite}. \quad (13)$$

By expanding the both sides of Eq. (13) as power series of  $t$  and equating the coefficients of like powers, we get

$$\begin{aligned} t : \quad & \sum_{j=1}^n \mu_j \Psi_j = X(1), \\ t^3 : \quad & \sum_{j=1}^n \mu_j \Psi_j^3 = -3!X(3), \\ t^5 : \quad & \sum_{j=1}^n \mu_j \Psi_j^5 = 5!X(5), \\ t^7 : \quad & \sum_{j=1}^n \mu_j \Psi_j^7 = -7!X(7), \\ & \dots \end{aligned} \quad (14)$$

In order to express  $\Phi_N(t)$  as an approximate periodic solution using two sines with two different amplitudes and two different arguments, we rewrite Eq. (13) at  $n = 2$ , as:

$$\Phi_7(t) = \sum_{j=1}^2 \mu_j \sin(\Psi_j t), \quad (15)$$

where the four unknowns  $\mu_1, \mu_2, \Psi_1$  and  $\Psi_2$  can be determined by solving the following system of non-linear algebraic equations:

$$\begin{aligned} \mu_1\Psi_1 + \mu_2\Psi_2 &= X(1), \\ \mu_1\Psi_1^3 + \mu_2\Psi_2^3 &= -3!X(3), \\ \mu_1\Psi_1^5 + \mu_2\Psi_2^5 &= 5!X(5), \\ \mu_1\Psi_1^7 + \mu_2\Psi_2^7 &= -7!X(7), \end{aligned} \tag{16}$$

while expressing  $\Phi_N(t)$  in terms of three sines, yields

$$\Phi_{11}(t) = \sum_{j=1}^3 \mu_j \sin(\Psi_j t). \tag{17}$$

In this case, the six unknowns  $\mu_1, \mu_2, \mu_3, \Psi_1, \Psi_2$  and  $\Psi_3$  can be determined by solving the following system of non-linear algebraic equations, numerically:

$$\begin{aligned} \mu_1\Psi_1 + \mu_2\Psi_2 + \mu_3\Psi_3 &= X(1), \\ \mu_1\Psi_1^3 + \mu_2\Psi_2^3 + \mu_3\Psi_3^3 &= -3!X(3), \\ \mu_1\Psi_1^5 + \mu_2\Psi_2^5 + \mu_3\Psi_3^5 &= 5!X(5), \\ \mu_1\Psi_1^7 + \mu_2\Psi_2^7 + \mu_3\Psi_3^7 &= -7!X(7), \\ \mu_1\Psi_1^9 + \mu_2\Psi_2^9 + \mu_3\Psi_3^9 &= 9!X(9), \\ \mu_1\Psi_1^{11} + \mu_2\Psi_2^{11} + \mu_3\Psi_3^{11} &= -11!X(11). \end{aligned} \tag{18}$$

It should be noted that, if the truncated series obtained through differential transformation method is expressed in mixed-powers of  $t$ , odd and even, then we use both of SAT and CAT techniques to deal with such truncated series. This point is indicated by an example of Van der Pol equation, in Section 6.

## 5 One-dimensional differential transform

In this section we discuss the application of the proposed CAT and SAT techniques to the periodic solutions of the Duffing equation:

$$\frac{d^2x}{dt^2} + x + \epsilon x^3 = 0, \tag{19}$$

under two different cases of initial conditions, as explained in the next subsections. As explained in the previous sections, the use of the proposed CAT and SAT techniques is based on obtaining the series solution of Eq. (19), under prescribed initial conditions, as a finite polynomial in even-powers of  $t$  or in odd-powers of  $t$ , respectively.

### 5.1 Case 1

Now, consider Eq. (19) under the following initial conditions

$$x(0) = a, \dot{x}(0) = 0. \tag{20}$$

The exact solution of Eqs. (19) and (20) has been resulted by Feng [21] as:

$$x = a \operatorname{cn}(\omega t, k^2), \quad \omega^2 = 1 + \epsilon a^2, \quad k^2 = \frac{\epsilon a^2}{2(1 + \epsilon a^2)}. \tag{21}$$

The validity and effectiveness of the proposed CAT is then checked by comparing the approximate periodic solution with the exact one at chosen values of  $a$  and  $\epsilon$ .

#### 5.1.1 Approximate periodic solution via CAT technique

In [22], the differential transformation method has been applied to Eq. (19) as

$$(k + 1)(k + 2)X(k + 2) + X(k) + \epsilon \sum_{s=0}^k \sum_{m=0}^{k-s} X(s)X(m)X(k - s - m) = 0, \tag{22}$$

with the transformed initial conditions

$$X(0) = a, \quad X(1) = 0. \tag{23}$$

The truncated series  $\Phi_6(t)$  is given by [22]:

$$\Phi_6(t) = a - a(1 + \epsilon a^2) \frac{t^2}{2!} + a(1 + \epsilon a^2)(1 + 3\epsilon a^2) \frac{t^4}{4!} - a(1 + \epsilon a^2)(1 + 24\epsilon a^2 + 27\epsilon^2 a^4) \frac{t^6}{6!}. \tag{24}$$

From the last equation, we can write

$$X(0) = a, \tag{25}$$

$$X(2) = \frac{-a(1 + \epsilon a^2)}{2!}, \tag{26}$$

$$X(4) = \frac{a(1 + \epsilon a^2)(1 + 3\epsilon a^2)}{4!}, \tag{27}$$

$$X(6) = \frac{-a(1 + \epsilon a^2)(1 + 24\epsilon a^2 + 27\epsilon^2 a^4)}{6!}, \tag{28}$$

Now, we can express  $\Phi_6(t)$  as an approximate periodic solution in the form

$$\Phi_6(t) = \sum_{j=1}^2 \lambda_j \cos(\Omega_j t). \tag{29}$$

By inserting  $X(0)$ ,  $X(2)$ ,  $X(4)$ ,  $X(6)$  presented above into system (10), we have a system of four non-linear algebraic equations. Using MATHEMATICA to solve this system analytically for  $\lambda_1$ ,  $\lambda_2$ ,  $\Omega_1$  and  $\Omega_2$ , we obtain

$$\begin{aligned} \lambda_1 &= \frac{a(4+5\epsilon a^2 + \sqrt{(2+3\epsilon a^2)(8+9\epsilon a^2)})}{2\sqrt{(2+3\epsilon a^2)(8+9\epsilon a^2)}}, & \Omega_1 &= \pm \sqrt{5 + 6\epsilon a^2 - \sqrt{(2 + 3\epsilon a^2)(8 + 9\epsilon a^2)}}, \\ \lambda_2 &= \frac{a(-4-5\epsilon a^2 + \sqrt{(2+3\epsilon a^2)(8+9\epsilon a^2)})}{2\sqrt{(2+3\epsilon a^2)(8+9\epsilon a^2)}}, & \Omega_2 &= \pm \sqrt{5 + 6\epsilon a^2 + \sqrt{(2 + 3\epsilon a^2)(8 + 9\epsilon a^2)}}. \end{aligned} \tag{30}$$

Therefore, we can write the approximate periodic solution for the Eqs. (19-20), as follows

$$\begin{aligned} x_{\text{approx}}(t) &= \left[ \frac{a(4+5\epsilon a^2 + \sqrt{(2+3\epsilon a^2)(8+9\epsilon a^2)})}{2\sqrt{(2+3\epsilon a^2)(8+9\epsilon a^2)}} \right] \times \\ &\quad \times \cos \left( \sqrt{5 + 6\epsilon a^2 - \sqrt{(2 + 3\epsilon a^2)(8 + 9\epsilon a^2)}} t \right) \\ &+ \left[ \frac{a(-4-5\epsilon a^2 + \sqrt{(2+3\epsilon a^2)(8+9\epsilon a^2)})}{2\sqrt{(2+3\epsilon a^2)(8+9\epsilon a^2)}} \right] \times \\ &\quad \times \cos \left( \sqrt{5 + 6\epsilon a^2 + \sqrt{(2 + 3\epsilon a^2)(8 + 9\epsilon a^2)}} t \right). \end{aligned} \tag{31}$$

In order to check the effectiveness of the Cosine-AT in finding accurate periodic solution for Eqs. (19-20), we compare our approximate periodic solution (31) with the exact one (21) at  $a = 0.1$ ,  $\epsilon = 0.1$ . Substituting  $a = 0.1$ ,  $\epsilon = 0.1$  into Eq. (31), we obtain the approximate periodic solution as:

$$x_{\text{approx}}(t) = 0.0999969 \cos(1.00037 t) + 3.12003 \times 10^{-6} \cos(3.00187 t). \tag{32}$$

Fig. 1 shows the curves of the exact and approximate periodic solutions at  $a = 0.1$ ,  $\epsilon = 0.1$ . As shown in this figure, the CAT approximate periodic solution is identical to the exact solution.

### 5.2 Case 2

In this case we hope firstly to obtain the exact solution of the Duffing equation subject to present initial conditions,

$$x(0) = 0, \quad \dot{x}(0) = \beta, \tag{33}$$

in terms of Jacobi-elliptic function  $sn$ . As analyzed in case 1, obtaining such exact solution is very useful to check the validity and effectiveness of SAT proposed in this work.

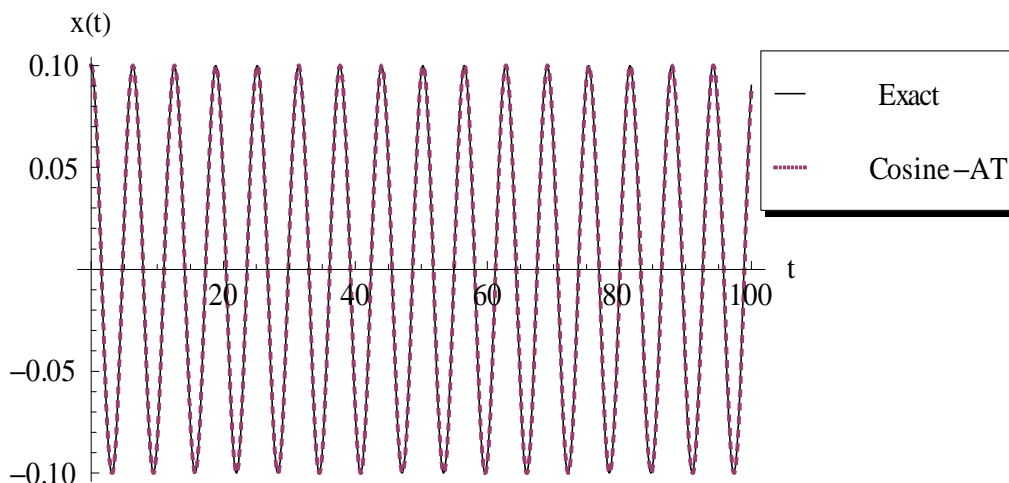


Figure 1: Comparison of the approximate periodic solution, Eq. (32), using CAT technique and the exact solution for the Duffing equation at  $\epsilon=0.1$  and  $a=0.1$ .

**5.2.1 Exact solution**

Assume an exact solution of Eq. (19) in the form of the Jacobi elliptic function  $sn$ :

$$x = \mu \operatorname{sn}(\sigma t, m^2), \tag{34}$$

where  $\sigma$  is the frequency of vibrations,  $\mu$  is the amplitude of vibrations and  $m^2$  is the modulus of the elliptic Jacobi function  $sn$ . Firstly, we note that the initial condition  $x(0) = 0$ , is already satisfied by the above assumption. Applying the second initial condition  $\dot{x}(0) = 0$ , yields  $\sigma\mu = \beta$ . Substituting (34) along with the result  $\sigma\mu = \beta$  into (19) and equating coefficients by the same order of function  $sn$ , we obtain the values of  $\sigma$  and  $m^2$  as:

$$\mu^2 = \frac{-1 + \sqrt{1 + 2\epsilon\beta^2}}{\epsilon}, \quad \sigma^2 = \frac{1}{2}(1 + \sqrt{1 + 2\epsilon\beta^2}), \quad m^2 = \frac{1 - \sqrt{1 + 2\epsilon\beta^2}}{1 + \sqrt{1 + 2\epsilon\beta^2}}. \tag{35}$$

**5.2.2 Approximate periodic solution via SAT technique**

On applying the same recurrence relation given by Eq. (22) with the transformed initial conditions

$$X(0) = 0, \quad X(1) = \beta, \tag{36}$$

we obtain

$$X(3) = -\frac{\beta}{3!}, \tag{37}$$

$$X(5) = \frac{\beta(1 - 6\epsilon\beta^2)}{5!}, \tag{38}$$

$$X(7) = -\frac{\beta(1 - 66\epsilon\beta^2)}{7!}, \tag{39}$$

$$X(9) = \frac{\beta(1 - 612\epsilon\beta^2 + 756\epsilon^2\beta^4)}{9!}, \tag{40}$$

$$X(11) = -\frac{\beta(1 - 5532\epsilon\beta^2 + 33156\epsilon^2\beta^4)}{11!}. \tag{41}$$

Now the truncated series solutions  $\Phi_7(t)$  and  $\Phi_{11}(t)$  are given by

$$\Phi_7(t) = \beta t - \frac{\beta t^3}{3!} + \frac{\beta(1 - 6\epsilon\beta^2)t^5}{5!} - \frac{\beta(1 - 66\epsilon\beta^2)t^7}{7!}, \tag{42}$$

and

$$\Phi_{11}(t) = \beta t - \frac{\beta t^3}{3!} + \frac{\beta(1-6\epsilon\beta^2)t^5}{5!} - \frac{\beta(1-66\epsilon\beta^2)t^7}{7!} + \frac{\beta(1-612\epsilon\beta^2+756\epsilon^2\beta^4)t^9}{9!} - \frac{\beta(1-5532\epsilon\beta^2+33156\epsilon^2\beta^4)t^{11}}{11!}. \tag{43}$$

If we expand the exact solution up to  $t^7$  and  $t^{11}$  we obtain the same truncated series solutions (42) and (43), respectively. By using the proposed SAT, we suppose that the approximate periodic solution  $\Phi_7(t)$  is expressed as

$$x_{\text{approx}}(t) = \mu_1 \sin(\Psi_1 t) + \mu_2 \sin(\Psi_2 t). \tag{44}$$

In order to find the values of  $\mu_1, \mu_2, \Psi_1$  and  $\Psi_2$ , we need only to insert  $X(1), X(3), X(5)$  and  $X(7)$  into system (16) to obtain a system of four non-linear algebraic equations, given by:

$$\begin{aligned} \mu_1 \Psi_1 + \mu_2 \Psi_2 &= \beta, \\ \mu_1 \Psi_1^3 + \mu_2 \Psi_2^3 &= \beta, \\ \mu_1 \Psi_1^5 + \mu_2 \Psi_2^5 &= \beta(1 - 6\epsilon\beta^2), \\ \mu_1 \Psi_1^7 + \mu_2 \Psi_2^7 &= \beta(1 - 66\epsilon\beta^2). \end{aligned} \tag{45}$$

In fact, we can solve the above system analytically, however, for simplicity we solve it numerically by using the NSolve command in MATHEMATICA, at  $\beta = 1, \epsilon = 0.3$ . By this, we obtain the following approximate periodic solution:

$$x_{\text{approx}}(t) = 0.928746 \sin(1.10982 t) - 0.010383 \sin(2.96113 t), \tag{46}$$

which is the same approximate periodic solution given in [20, Eq. (3.23)] which obtained by using Laplace transform and Padé approximate [4/4]. In Fig. 2, the approximate (46) and exact solutions are plotted. It can be concluded that the SAT solution is very close to the exact solution. In order to examine the assumption that expressing the truncated series in more sines with different amplitudes and arguments leads to more accurate numerical periodic solution, we use Eq. (17) with the following system at the same values  $\beta = 1, \epsilon = 0.3$ :

$$\begin{aligned} \mu_1 \Psi_1 + \mu_2 \Psi_2 + \mu_3 \Psi_3 &= \beta, \\ \mu_1 \Psi_1^3 + \mu_2 \Psi_2^3 + \mu_3 \Psi_3^3 &= \beta, \\ \mu_1 \Psi_1^5 + \mu_2 \Psi_2^5 + \mu_3 \Psi_3^5 &= \beta(1 - 6\epsilon\beta^2), \\ \mu_1 \Psi_1^7 + \mu_2 \Psi_2^7 + \mu_3 \Psi_3^7 &= \beta(1 - 66\epsilon\beta^2), \\ \mu_1 \Psi_1^9 + \mu_2 \Psi_2^9 + \mu_3 \Psi_3^9 &= \beta(1 - 612\epsilon\beta^2 + 756\epsilon^2\beta^4), \\ \mu_1 \Psi_1^{11} + \mu_2 \Psi_2^{11} + \mu_3 \Psi_3^{11} &= \beta(1 - 5532\epsilon\beta^2 + 33156\epsilon^2\beta^4), \end{aligned} \tag{47}$$

to obtain the approximate periodic solution:

$$x_{\text{approx}}(t) = 0.933608 \sin(1.09232 t) - 0.006132 \sin(3.39027 t) + 0.000210 \sin(4.72365 t). \tag{48}$$

This approximate periodic solution is plotted in Fig. 3 with the exact one. As showed in this figure, the result is excellent in more wider range than [20, Fig. 4], without any need to Padé approximates or Laplace transform.

## 6 Application to Van der Pol equation

Consider the following Van der Pol equation [20]

$$\frac{d^2 x}{dt^2} + x = \epsilon(1 - x^2)\dot{x}, \tag{49}$$

under the initial conditions

$$x(0) = 0, \dot{x}(0) = 2. \tag{50}$$

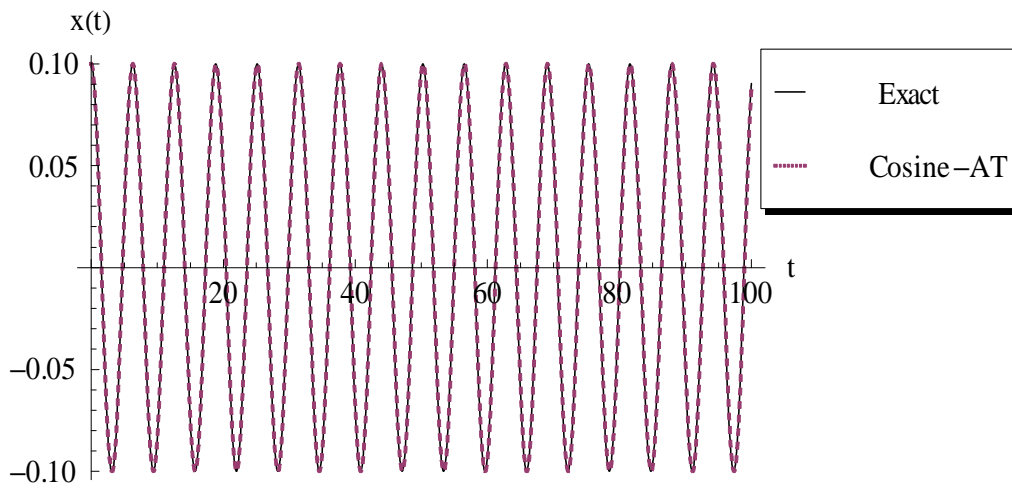


Figure 2: Comparison of the approximate periodic solution, Eq. (32), using CAT technique and the exact solution for the Duffing equation at  $\epsilon=0.1$  and  $a=0.1$ .

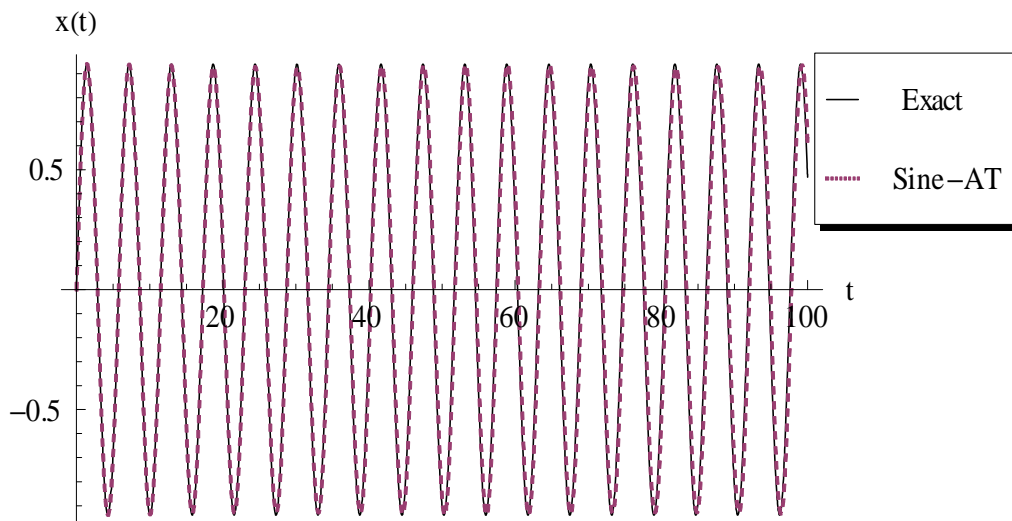


Figure 3: Comparison of the approximate periodic solution, Eq. (48), using SAT technique and the exact solution for the Duffing equation at  $\epsilon=0.3$  and  $\beta=1$ .



By using differential transformation method, the following series solution is obtained by [20] as:

$$\Phi_7(t) = 2 \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \right) + \epsilon \left( t^2 - \frac{5t^4}{6} + \frac{91t^6}{360} \right). \quad (51)$$

In this example, the truncated series solution is expressed in mixed-powers of  $t$ , so, we have to use both of SAT and CAT techniques. Firstly, from this truncated series solution, we can get

$$X(0) = 0, X(2) = \epsilon, X(4) = -\frac{5\epsilon}{6}, X(6) = \frac{91\epsilon}{360}, \quad (52)$$

$$X(1) = 2, X(3) = -\frac{1}{3}, X(5) = \frac{2}{5!}, X(7) = -\frac{2}{7!}. \quad (53)$$

Now, suppose an approximate periodic solution in the form

$$x = \lambda_1 \cos(\Omega_1 t) + \lambda_2 \cos(\Omega_2 t) + \mu_1 \sin(\Psi_1 t) + \mu_2 \sin(\Psi_2 t). \quad (54)$$

By inserting  $X(0)$ ,  $X(2)$ ,  $X(4)$  and  $X(6)$  into system (10), and inserting  $X(1)$ ,  $X(3)$ ,  $X(5)$  and  $X(7)$  into system (18) and solving the resulting systems analytically for the values of  $\lambda_i$ ,  $\Omega_i$ ,  $\mu_i$ ,  $\Psi_i$ ,  $i = 1, 2$ , we obtain

$$\lambda_1 = \frac{\epsilon}{4}, \lambda_2 = -\frac{\epsilon}{4}, \Omega_1 = 1, \Omega_2 = -3, \mu_1 = 0, \mu_2 = 2, \Psi_2 = 1. \quad (55)$$

Hence, we have the following approximate periodic solution for the Van der Pol equation:

$$x_{\text{approx}}(t) = 2\sin(t) + \frac{\epsilon}{4} [\cos(t) - \cos(3t)]. \quad (56)$$

Now, it should be noted that the approximate periodic solution given by Eq. (56), which obtained by using our proposed SAT and CAT techniques, is the same one obtained in [20, Eq. (3.8)] by using Padé approximants and Laplace transform. In [20, Eq. (3.8)], the approximate periodic solution is obtained as:

$$x = 2\sin(t) + \epsilon \cos(t) \sin^2(t), \quad (57)$$

which can be written as:

$$x = 2\sin(t) + \frac{1}{2}\epsilon \sin(t) \sin(2t). \quad (58)$$

On using the trigonometric identity:  $\sin\theta \sin\phi = \frac{1}{2}[\cos(\theta - \phi) - \cos(\theta + \phi)]$  into Eq. (58), we obtain our result given by Eq. (56).

**Remark 1** It should be noted that the present technique works also well on applying on the other types of nonlinear oscillator equations including the pendulum one,  $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$ , where Chang and Chang [17] applied the DTM on the nonlinear term  $\sin\theta$ .

## 7 Conclusion

In this research, we have proposed a new aftertreatment technique to deal with the truncated series obtained via differential transformation method. The proposed Sine-AT and Cosine-AT has been applied successfully to the Duffing equation with two different initial conditions and Van der Pol equation. The approximate periodic solutions obtained in this paper are found to be in excellent agreement with the exact solutions. Furthermore, the results obtained in this research are found to be in good agreement with those obtained previously by the classical AT-technique [18, 19], without any need to Padé approximants or Laplace transform.

## References

- [1] G. Adomian. Solving Frontier Problems of Physics: The Decomposition Method. *Kluwer. Acad.: Boston*(1994).
- [2] J. K. Zhou. Differential Transformation and its Applications for Electrical Circuits. *Huazhong University Press: Wuhan, China*(1986).

- [3] A. Ebaid. A new numerical solution for the MHD peristaltic flow of a bio-fluid with variable viscosity in a circular cylindrical tube via Adomian decomposition method. *Phys. Lett. A.*, 372:(2008),5321-5328.
- [4] N. T. Eldabe, E. M. Elghazy and A. Ebaid. Closed form solution to a second order boundary value problem and its application in fluid mechanics. *Phys. Lett. A.*, 363:(2007), 257-259.
- [5] N. T. Shawagfeh. Analytic approximate solution for a nonlinear oscillator equation. *Computers Math. Applic.*, 31:(1996), 135-141.
- [6] S. H. Ho and C. K. Chen. Analysis of general elastically end restrained non-uniform beams using differential transform. *Appl. Math. Modell.*, 22:(1998),219-234.
- [7] C. K. Chen and S. H. Ho. Transverse vibration of a rotating twisted Timoshenko beams under axial loading using differential transform. *Int. J. Mechanical Sciences.*, 41:(1999), 1339-1356.
- [8] M. J. Jang, C. L. Chen and Y. C. Liy. On solving the initial-value problems using the differential transformation method. *Appl. Math. Comput.*, 115:(2000), 145-160.
- [9] M. Köksal and S. Herdem. Analysis of nonlinear circuits by using differential Taylor transform. *Computers and Electrical Engineering*, 28:(2002), 513-525.
- [10] I. H. A-H Hassan. Different applications for the differential transformation in the differential equations. *Appl. Math. Comput.*, 129:(2002), 183-201.
- [11] A. Arikoglu and I. Ozkol. Solution of boundary value problems for integro-differential equations by using differential transform method. *Appl. Math. Comput.*, 168:(2005),1145-1158.
- [12] A. S. V. Ravi Kanth and K. Aruna. Solution of singular two-point boundary value problems using differential transformation method. *Phys. Lett. A.*, 372:(2008), 4671-4673.
- [13] C. K. Chen and S. H. Ho. Solving partial differential equations by two-dimensional differential transform method. *Appl. Math. Comput.*, 41:(1999),171-179.
- [14] M. J. Jang, C. L. Chen and Y. C. Liy. Two-dimensional differential transform for partial differential equations. *Appl. Math. Comput.*, 121:(2001),261-270.
- [15] F. Ayaz. On two-dimensional differential transform method. *Appl. Math. Comput.*, 143:(2003),361-374.
- [16] F. Ayaz. Solutions of the system of differential equations by differential transform method. *Appl. Math. Comput.*, 147:(2004), 547-567.
- [17] S. H. Chang and I. L. Chang. A new algorithm for calculating one-dimensional differential transform of nonlinear functions. *Appl. Math. Comput.*, 195:(2008), 799-808.
- [18] S. N. Venkatarangan and K. Rajakshmi. A modification of adomian's solution for nonlinear oscillatory systems. *Comput. Math. Appl.*, 29:(1995), 67-73.
- [19] Y. C. Jiao, Y. Yamamoto, C. Dang and Y. Hao. An aftertreatment technique for improving the accuracy of Adomian's decomposition method. *Comput. Math. Appl.*, 43:(2002),783-798.
- [20] S. Momani and V.S. Ertürk. Solutions of non-linear oscillators by the modified differential transform method. *Computers and Mathematics with Applications*, 55:(2008), 833-842.
- [21] Z. Feng. Duffing's equation and its applications to Hirota equation. *Phys. Lett. A.*, 317:(2003),115-119.
- [22] M. El-Shahed. Application of differential transform method to non-linear oscillatory systems. *Communic. Nonlin. Scien. Numer. Simul.*, 13:(2008),1714-1720.