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Numerical Solutions of Generalized Drinfeld-Sokolov Equations Using the Homotopy Analysis Method

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Abstract: In this paper, a system of generalized Drinfeld-Sokolov (gDS) equation is solved by means of an analytic technique, namely the Homotopy analysis method (Shortly HAM). Comparisons are made among the Adomian decomposition method, the exact solution and the homotopy analysis method. The results reveal that the proposed method is very effective and simple.

Keywords: Drinfeld-Sokolov equations; Homotopy analysis method; Adomian decomposition method; numerical solution

1 Introduction

In 1992, Liao[11] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (HAM)[11–14]. This method has been successfully applied to solve many types of nonlinear problems [9, 15–18]. The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy, and requires large computer power and time. HAM method is better since it does not involve discretization of the variables hence is free from rounding off errors and does not require large computer memory or time.

In this paper, we propose homotopy analysis method to solve generalized DrinfeldSokolov (gDS) equation. Comparisons are made among the Adomian decomposition method, the exact solution and the proposed method.

2 Basic idea of HAM

To illustrate of basic idea of HAM we consider the following differential equation

$$\mathcal{N}[u(\tau)] = 0, \tag{1}$$

where $\mathcal{N}$ is a nonlinear operator, τ denotes independent variable, $u(\tau)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [12] constructs the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] = p\,\hbar\mathcal{H}(\tau)\mathcal{N}[\phi(\tau;p)], \tag{2}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $\mathcal{H}(\tau) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, $u(\tau;p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\phi(\tau;0) = u_0(\tau), \phi(\tau;1) = u(\tau), \tag{3}$$

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respectively. Thus, as p increases from 0 to 1, the solution $u(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $u(\tau;p)$ in Taylor series with respect to p, we have

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m, \tag{4}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m}|_{p=0}. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (4) converges at $p = 1$, then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{6}$$

which must be one of solutions of original nonlinear equation, as proved by[12]. As $\hbar = -1$ and $\mathcal{H}(\tau) = 1$, Eq. (2) becomes

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] + p\,\mathcal{N}[\phi(\tau;p)] = 0, \tag{7}$$

which is used mostly in the homotopy perturbation method[6], where as the solution obtained directly, without using Taylor series [5, 7, 8, 10]. According to the definition (5), the governing equation can be deduced from the zero-order deformation equation (2). Define the vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\}.$$

Differentiating equation (2) m-times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar\,\mathcal{H}(\tau)\mathcal{R}_m(\vec{u}_{m-1}), \tag{8}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(\tau;p)]}{\partial p^{m-1}}|_{p=0}. \tag{9}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{10}$$

It should be emphasized that $u_m(\tau)$ for $m \geqslant 1$ is governed by the linear equation (8) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab. For the convergence of the above method we refer the reader to Liao's work [12] . If Eq. (1) admits unique solution, then this method will produce the unique solution. If equation (1) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

3 Application

In order to assess the advantages and the accuracy of homotopy analysis method for solving nonlinear equations, we will consider the following example.

Example 1 *We consider the* **generalized Drinfeld-Sokolov** *equations (gDS) equations in two space dimension:*

$$u_t + u_{xxx} - 6uu_x - 6(v^\alpha)_x = 0, \qquad v_t - 2v_{xxx} + 6uv_x = 0. \tag{11}$$

where α is a constant. The multiple soliton-like solution of Eqs. (11) when $\alpha = 2$ obtained in[19] as follows:

$$u(x,t) = \frac{-b_1^2 - 4k^4}{4k^2} + 2k^2\tanh^2(kx + \frac{3b_1^2+4k^4}{2k}t), \quad v(x,t) = b_1\tanh(kx + \frac{3b_1^2+4k^4}{2k}t). \tag{12}$$

To solve the Eqs. (11) by means of homotopy analysis method, we choose the linear operator

$$\mathcal{L}_i[\phi_i(x,t;p)] = \frac{\partial \phi_i(x,t;p)}{\partial t}, \quad i = 1,2. \tag{13}$$

The inverse operator $\mathcal{L}_i^{-1}$ is given by

$$\mathcal{L}_i^{-1}(\cdot) = \int_0^t (\cdot) ds, \quad i = 1,2. \tag{14}$$

We now define a nonlinear operators as:

$$\begin{aligned} \mathcal{N}_1[\phi_1,\phi_2] &= \frac{\partial \phi_1(x,t;p)}{\partial t} + \frac{\partial^3 \phi_1(x,t;p)}{\partial x^3} - 6\phi_1(x,t;p)\frac{\partial \phi_1(x,t;p)}{\partial x} - 6\frac{\partial \phi_2^\alpha(x,t;p)}{\partial x}, \\ \mathcal{N}_2[\phi_1,\phi_2] &= \frac{\partial \phi_2(x,t;p)}{\partial t} - 2\frac{\partial^3 \phi_2(x,t;p)}{\partial x^3} + 6\phi_1(x,t;p)\frac{\partial \phi_2(x,t;p)}{\partial x}. \end{aligned} \tag{15}$$

Using above definition, we construct the zeroth-order deformation equations:

$$\begin{aligned} (1-p)\mathcal{L}_1[\phi_1(x,t;p) - u_0(x,t)] = p\,\hbar_1\mathcal{N}_1[\phi_1,\phi_2], \\ (1-p)\mathcal{L}_2[\phi_2(x,t;p) - v_0(x,t)] = p\,\hbar_2\mathcal{N}_2[\phi_1,\phi_2]. \end{aligned} \tag{16}$$

For$p = 0$ and $p = 1$, we can write

$$\begin{aligned} \phi_1(x,t;0) = u_0(x,t), \quad \phi_2(x,t;0) = v_0(x,t), \\ \phi_1(x,t;1) = u(x,t), \quad \phi_2(x,t;1) = v(x,t). \end{aligned} \tag{17}$$

Thus, we obtain the mth-order deformation equations:

$$\begin{aligned} \mathcal{L}_1[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar_1\mathcal{H}_1(x,t)\mathcal{R}_{1,m}(\vec{u}_{m-1}), \\ \mathcal{L}_2[v_m(x,t) - \chi_m v_{m-1}(x,t)] = \hbar_2\mathcal{H}_2(x,t)\mathcal{R}_{2,m}(\vec{v}_{m-1}), \quad (m \geq 1), \end{aligned} \tag{18}$$

With initial condutions:

$$\begin{aligned} u_m(x,0) = 0, \qquad (u_m)_t(x,0) = 0, \\ v_m(x,0) = 0, \qquad (v_m)_t(x,0) = 0. \end{aligned} \tag{19}$$

When $\alpha = 2$ we have:

$$\begin{aligned} \mathcal{R}_{1,m}(\vec{u}_{m-1}) &= \frac{\partial u_{m-1}}{\partial t} + \frac{\partial^3 u_{m-1}}{\partial x^3} - 6\sum_{i=0}^{m-1} u_i \frac{\partial u_{m-1-i}}{\partial x} - 12\sum_{i=0}^{m-1} v_i \frac{\partial v_{m-1-i}}{\partial x}, \\ \mathcal{R}_{2,m}(\vec{v}_{m-1}) &= \frac{\partial v_{m-1}}{\partial t} - 2\frac{\partial^3 v_{m-1}}{\partial x^3} + 6\sum_{i=0}^{m-1} u_i \frac{\partial v_{m-1-i}}{\partial x}. \end{aligned} \tag{20}$$

Now the solution of the mth-order deformation equations (18-19)

$$\begin{aligned} u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar_1\mathcal{H}_1(x,t)\mathcal{L}_1^{-1}[\mathcal{R}_{1,m}(\vec{u}_{m-1})], \\ v_m(x,t) = \chi_m v_{m-1}(x,t) + \hbar_2\mathcal{H}_2(x,t)\mathcal{L}_2^{-1}[\mathcal{R}_{2,m}(\vec{v}_{m-1})]. \end{aligned} \tag{21}$$

We start with an initial approximation $u_0(x,t) = \frac{-b_1^2-4k^4}{4k^2} + 2k^2\tanh^2(kx)$, $v_0(x,t) = b_1\tanh(kx)$, we can obtain directly the other components as:

$$\begin{aligned}
u_1(x,t) &= h_1\left[(-32k^5(1-\tanh^2(kx))^2\tanh(kx)t + 16k^5t\tanh^3(kx)(1-\tanh^2(kx))\right.\\
&- 24k^3t((-\frac{1}{4}b_1^2 - k^4)k^2 + 2k^2\tanh^2(kx))(1-\tanh^2(kx))\tanh(kx)\\
&\left.- 12b_1^2\tanh(kx)(1-\tanh^2(kx)kt\right],\\
u_2(x,t) &= \frac{1}{2}h_1tk\left(-48h_1k^8\tanh^3(kx) + 24h_1b_1^2\tanh^3(kx) - 24h_1b_1^2\tanh(kx)\right.\\
&+48h_1k^8\tanh(kx) - 64h_1k^3\tanh(kx) + 64h_1k^4\tanh^3(kx)\\
&-504th_1k^7\tanh^4(kx)b_1^2 + 288th_1b_1^2k^3\tanh^4(kx) - 54th_1k^3b_1^4\tanh^4(kx) - \cdots\\
&\vdots\\
v_1(x,t) &= h_2k^3t\left[4b_1(1-\tanh^2(kx))^2 - 8k^3tb_1\tanh^2(kx)(1-\tanh^2(kx))\right.\\
&\left.+6((-\frac{1}{4}b_1^2 - k^4)k^2 + 2k^2\tanh^2(kx))b_1(1-\tanh^2(kx))kt\right]\\
v_2(x,t) &= \frac{1}{4}h_2tb_1k^2\left(16k - 16\tanh^2(kx) - 24k^5 - 6b_1^2k - 6h_2b_1^2k - 24h_2k^5\right.\\
&+16h_2k + 9th_2k^4b_1^4\tanh^3(kx) - 192th_2b_1^2k^4\tanh^3(kx) + 72th_2b_1^2k^4\tanh^5(kx)\\
&-9th_2k^4b_1^4\tanh(kx) + 120th_2b_1^2k^4\tanh(kx) + 72th_1b_1^2k^4\tanh(kx)\cdots\\
&\vdots
\end{aligned}$$

For numerical comparisons purposes, we construct the solution $u(x,t)$ and $v(x,t)$

$$\lim_{n\to\infty}\phi_n = u(x,t), \qquad \lim_{n\to\infty}\psi_n = v(x,t),$$

where $\phi_n = \sum_{k=0}^{n} u_k(x,t)$ and $\psi_n = \sum_{k=0}^{n} v_k(x,t), \quad n \geq 0.$

4 Numerical results

We now present a numerical results in which $b_1 = 0.001,\ \ k = 0.01,\ \hbar_1 = -10^{-5},\ \ \hbar_2 = -10^{-9},\ \ \mathcal{H}_1(x,t) = \mathcal{H}_2(x,t) = 1,\ \ n = 4$. In tables (1-4) we have presented error of approximate solutions by 5th-order HAM for different value of $\hbar$.

Table.1: The comparison of the results of the HAM($\hbar_1 = -10^{-5}$,$\hbar_2 = -10^{-9}$) with the analytical solution $u(x,t)$(12)

t_i/x_i	0.2	0.4	0.6	0.8	1.0
0.2	4.6083×10^{-16}	9.2164×10^{-16}	1.3824×10^{-15}	1.8432×10^{-15}	2.3038×10^{-15}
0.4	9.2166×10^{-16}	1.8433×10^{-15}	2.7649×10^{-15}	3.6863×10^{-15}	4.6077×10^{-15}
0.6	1.3825×10^{-15}	2.7649×10^{-15}	4.1473×10^{-15}	5.5295×10^{-15}	6.9115×10^{-15}
0.8	1.8433×10^{-15}	3.6866×10^{-15}	5.5297×10^{-15}	7.3727×10^{-15}	9.2154×10^{-15}
1.0	2.3042×10^{-15}	4.6082×10^{-15}	6.9122×10^{-15}	9.2159×10^{-15}	1.1519×10^{-14}

Table.2: The comparison of the results of the the ADM($\hbar_1 = \hbar_2 = -1$) with the analytical solution $u(x,t)$(12)

t_i/x_i	0.2	0.4	0.6	0.8	1.0
0.2	4.9287×10^{-11}	9.8563×10^{-11}	1.4784×10^{-10}	1.9710×10^{-10}	2.4637×10^{-10}
0.4	9.8595×10^{-11}	1.9715×10^{-10}	2.9569×10^{-10}	3.9423×10^{-10}	4.9275×10^{-10}
0.6	1.4792×10^{-10}	2.9575×10^{-10}	4.4357×10^{-10}	5.9137×10^{-10}	7.3916×10^{-10}
0.8	1.9727×10^{-10}	3.9437×10^{-10}	5.9146×10^{-10}	7.8854×10^{-10}	9.8558×10^{-10}
1.0	2.4664×10^{-10}	4.9302×10^{-10}	7.3938×10^{-10}	9.8572×10^{-10}	1.2320×10^{-09}

Table.3: The comparison of the results of the HAM($\hbar_1 = -10^{-5}$,$\hbar_2 = -10^{-9}$) with the analytical solution $v(x,t)$(12)

t_i/x_i	0.2	0.4	0.6	0.8	1.0
0.2	3.0400×10^{-12}	3.0400×10^{-12}	3.0399×10^{-12}	3.0398×10^{-12}	3.0397×10^{-12}
0.4	6.0800×10^{-12}	6.0799×10^{-12}	6.0798×10^{-12}	6.0796×10^{-12}	6.0794×10^{-12}
0.6	9.1200×10^{-12}	9.1199×10^{-12}	9.1197×10^{-12}	9.1194×10^{-12}	9.1191×10^{-12}
0.8	1.2160×10^{-11}	1.2160×10^{-11}	1.2160×10^{-11}	1.2159×10^{-11}	1.2159×10^{-11}
1.0	1.5200×10^{-11}	1.5200×10^{-11}	1.5199×10^{-11}	1.5199×10^{-11}	1.5198×10^{-11}

Table.4: The comparison of the results of the the ADM($\hbar_1 = \hbar_2 = -1$) with the analytical solution $v(x,t)$(12)

t_i/x_i	0.2	0.4	0.6	0.8	1.0
0.2	8.0304×10^{-10}	8.0303×10^{-10}	8.0301×10^{-10}	8.0299×10^{-10}	8.0296×10^{-10}
0.4	1.6061×10^{-09}	1.6061×10^{-09}	1.6060×10^{-09}	1.6060×10^{-09}	1.6059×10^{-09}
0.6	2.4091×10^{-09}	2.4091×10^{-09}	2.4090×10^{-09}	2.4090×10^{-09}	2.4089×10^{-09}
0.8	3.2122×10^{-09}	3.2121×10^{-09}	3.2121×10^{-09}	3.2120×10^{-09}	3.2119×10^{-09}
1.0	4.0152×10^{-09}	4.0151×10^{-09}	4.0151×10^{-09}	4.0150×10^{-09}	4.0148×10^{-09}

Numerical approximations show a high degree of accuracy and in most cases ϕ_n and ψ_n, the n-term approximations for u and v, respectively are accurate for quite low values of n. The numerical results we obtained justify the advantage of this methodology. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series.

In figs (1- 4) we have presented approximate and exact solutions by 4th-order HAM for $\hbar_1 = -10^{-5}$,$\hbar_2 = -10^{-9}$.

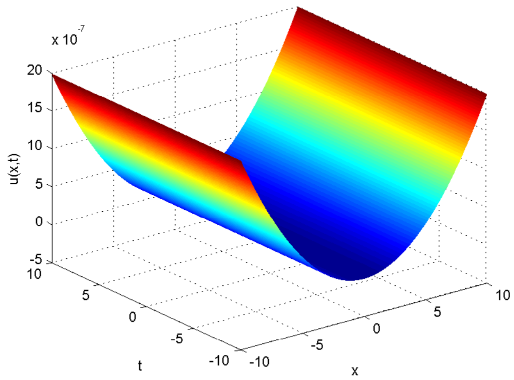

Fig1. (The 4th-order HAM approximate u(x,t))

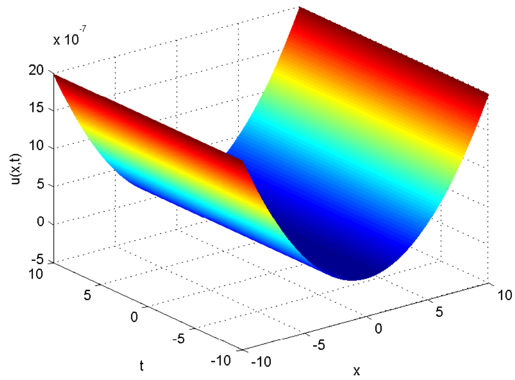

Fig2. (The multiple soliton-like solution u(x,t))

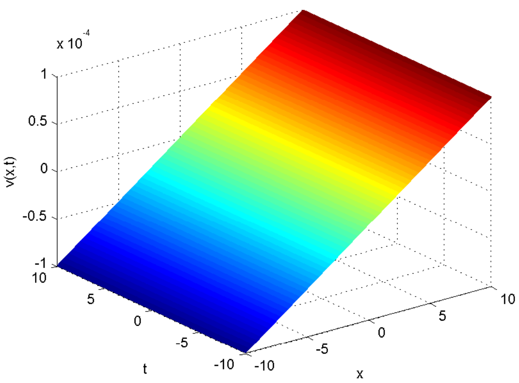

Fig3. (The 4th-order HAM approximate v(x,t))

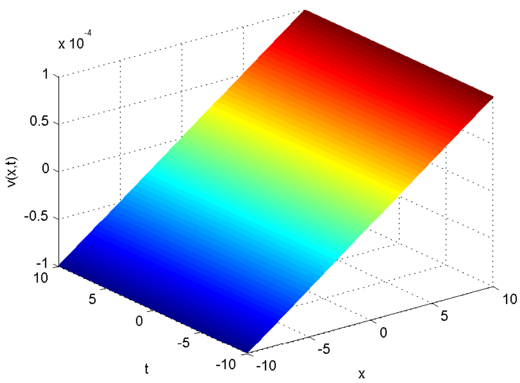

Fig4. (The multiple soliton-like solution v(x,t))

5 Conclusion

In this paper, we have successfully developed HAM for solving generalized DrinfeldSokolov equation. It is apparently seen that HAM is a very powerful and efficient technique in finding analytical solutions for wide classes of nonlinear systems. It is worth pointing out that this method presents a rapid convergence for the solutions. In conclusion, HAM provides accurate numerical solution for nonlinear problems in comparison with the Adomian decomposition method. They also do not require large computer memory and discretization of the variables t and x. The results show that HAM is powerful mathematical tool for solving linear partial differential equations. Matlab has been used for computations in this paper.

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