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Solution of an Initial Boundary Value Problem for Non-Planar Burgers Equation Using Hermite Interpolation

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Abstract: We study an initial boundary value problem for the non-planar Burgers equation using Hermite interpolants. Numerical solution and solutions obtained by Hermite interpolation are compared and are found to be in good agreement.

Keywords: Hermite interpolation; Non-planar Burgers equation; Initial boundary value problems

1 Introduction

In this paper, we study an initial boundary value problem (IBVP) for the non-planar Burgers equation, namely,

$$u_t + u^\alpha u_x + \frac{ju}{2(t+1)} = \epsilon u_{xx}, \;\; 0 < x < 1, \; t > 0, \tag{1}$$

$$u(x, 0) = u_0(x), \;\; 0 \leq x \leq 1, \tag{2}$$

$$u(0, t) = a, \;\; u(1, t) = b, \;\; t \geq 0, \tag{3}$$

where $\epsilon > 0$ is small, $\alpha \geq 1$ is an integer and $j > 0$ are parameters. Further, a and b are non-negative constants. We assume throughout that $u_0(x) \geq 0$ is sufficiently smooth on $[0, 1]$. Equation (1) has applications in nonlinear acoustics (see, for example, Enflo and Hedberg [2]). Following Grundy [3], we construct Hermite interpolants to approximate the solution of the IBVP (1)–(3). Then, we compare the Hermite interpolants with a numerical solution of the IBVP (1)–(3) obtained by a finite difference scheme due to Dawson [1]. Based on excellent agreement between the Hermite interpolants and numerical solution, we may conclude that a suitable Hermite interpolant solution is a good approximation to the solution of the IBVP (1)–(3) for all time.

Now, we define Hermite interpolants for a function (see, for example, Grundy [3]). Let $f(x)$ be a sufficiently smooth function defined on $[0, 1]$. Further, let $f^{(r)}(x)$ be the r^{th} derivative of $f(x)$; $f^{(r)}(0)$ and $f^{(r)}(1)$ are known for $r = 0, 1, ..., n$ for some positive integer n. Then, n^{th} order Hermite interpolant of $f(x)$, denoted by $p_n(x)$, is written as

$$p_n(x) = \sum_{r=0}^{n} \{f^{(r)}(0)Q_r^n(x) + (-1)^r f^{(r)}(1)Q_r^n(1 - x)\}, \;\; x \in [0, 1], \tag{4}$$

where $Q_r^n(x)$ is a polynomial of degree $2n + 1$ on $[0, 1]$, and is given by

$$Q_r^n(x) = \frac{x^r}{r!}(1 - x)^{n+1} \sum_{s=0}^{n-r} \frac{(n + s)!}{s!n!} x^s. \tag{5}$$

Thus, a Hermite interpolant approximates a function over an interval by making use of the values of the function and a certain number of its derivatives at the end points of the interval. The error in approximating the function f by the Hermite interpolant p_n on $[0, 1]$ is given by

$$f(x) - p_n(x) = (-1)^{n+1}x^{n+1}(1 - x)^{n+1}f^{(2n+2)}(\xi)/(2n + 2)!, \tag{6}$$

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for some $\xi \in (0,1)$ and $f^{(2n+2)}$ is assumed to be continuous in $(0,1)$.

The Hermite interpolation polynomial given by (4)–(5) is discussed in detail by Phillips [8]. Grundy and Phillips [6] exemplified the application of Hermite interpolants to estimate the initial values for solving a boundary value problem posed for ODEs of the form $y''(x) = f(x,y)$ on $[0,1]$. We may refer to Lanczos [7] and Grundy ([4], [5]) for a related study.

Sachdev and his collaborators (see [9], [10]) have studied the large time behaviour of periodic solutions of some generalized Burgers equations, including the non-planar Burgers equation, using a perturbative technique.

In the next section, we approximate the solution of the IBVP (1)–(3) by a suitable Hermite interpolant and then present a comparison of the Hermite interpolants with the numerical solution of the IBVP (1)–(3). Finally section 3 presents the conclusions.

2 Hermite interpolant solution of IBVP (1)–(3)

In this section, we find Hermite interpolants p_2, p_3, p_4 for approximating the solution of the initial boundary value problem (1)–(3) and compare them with numerical solution of the IBVP (1)–(3). This approximation is valid for all time t. The accuracy of the approximation depends on the order of the Hermite interpolant and the compatibility of the initial and boundary conditions with the given PDE.

Rewrite (1) as

$$u_{xx} = \frac{1}{\epsilon}\left[u_t + u^\alpha u_x + \frac{ju}{2(t+1)}\right].$$

We make use of the Green's function for the operator $\frac{\partial^2}{\partial x^2}$ subject to homogeneous Dirichlet boundary conditions at $x = 0$ and $x = 1$ to arrive at the following integro-differential equation

$$u(x,t) = a + (b-a)x + \frac{1}{\epsilon}\int_0^1 K(x,s)\left(u_t + u^\alpha u_s + \frac{ju}{2(t+1)}\right)ds, \tag{1}$$

where $K(x,s) = \begin{cases} s(x-1), & 0 \le s \le x \\ x(s-1), & x \le s \le 1 \end{cases}$. The first two terms on the right hand side of (1) are due to the nonhomogeneous boundary conditions (see (3)) at $x = 0$ and $x = 1$. Further, u_t and u_s in (1) are partial derivatives of $u(s,t)$ with respect to t and s respectively. Performing an integration by parts in (1), we arrive at

$$\begin{aligned} u(x,t) &= a + (b-a)x + \frac{1}{\epsilon}\left[\int_0^1 K(x,s)\left(u_t + \frac{ju}{2(t+1)}\right)ds \right. \\ &\quad \left. - \int_0^x (x-1)\frac{u^{\alpha+1}}{\alpha+1}ds - \int_x^1 x\frac{u^{\alpha+1}}{\alpha+1}ds\right]. \end{aligned} \tag{2}$$

Differentiating (2) with respect to x, we get

$$\begin{aligned} u_x(x,t) &= \frac{1}{\epsilon}\left[\int_0^x s\left(u_t + \frac{ju}{2(t+1)}\right)ds + \int_x^1 (s-1)\left(u_t + \frac{ju}{2(t+1)}\right)ds \right. \\ &\quad \left. - \int_0^1 \frac{u^{\alpha+1}}{\alpha+1}ds + \frac{u^{\alpha+1}}{\alpha+1}\right] + (b-a). \end{aligned} \tag{3}$$

Let

$$u_x(0,t) = V_0(t) \text{ and } u_x(1,t) = V_1(t). \tag{4}$$

Following (4) and (5), we write the Hermite interpolant approximation to $u(x,t)$ as

$$p_n(x,t) = \sum_{r=0}^{n}\{u_{x,r}(0,t)Q_r^n(x) + (-1)^r u_{x,r}(1,t)Q_r^n(1-x)\}, \quad x \in [0,1], \tag{5}$$

where $Q_r^n(x)$ is as in (5) and $u_{x,r}$ is the r^{th} partial derivative of u with respect to x. It may be noted here that $u_{x,r}(0,t)$ and $u_{x,r}(1,t)$ can be expressed in terms of $V_0(t)$, $V_1(t)$ and their derivatives.

We find the Hermite interpolants p_2, p_3 and p_4 and compare with numerical solution of the IBVP (1)–(3) for specific initial and boundary conditions. For this purpose, we need to find $u_{x,r}(0,t)$ and $u_{x,r}(1,t)$, $r = 1,2,3,4$ (see (5)). We give below derivation of expressions for $u_{x,r}(0,t)$ and $u_{x,r}(1,t)$ in terms of $V_0(t)$, $V_1(t)$ and their derivatives. Clearly from (3) and (4)

$$\begin{aligned} &u_{x,0}(0,t) = u(0,t) = a, \;\; u_{x,0}(1,t) = u(1,t) = b, \\ &u_{x,1}(0,t) = V_0(t), \;\; u_{x,1}(1,t) = V_1(t). \end{aligned}$$

Making use of the equation (1) and the boundary conditions (3), we get

$$u_{x,2}(0,t) = \frac{1}{\epsilon}\left(a^\alpha V_0 + \frac{aj}{2(t+1)}\right), \;\; u_{x,2}(1,t) = \frac{1}{\epsilon}\left(b^\alpha V_1 + \frac{bj}{2(t+1)}\right). \tag{6}$$

We illustrate the computation of $u_{x,3}(0,t)$:

$$\begin{aligned} u_{x,3}(0,t) &= \frac{1}{\epsilon}\left(u_t + u^\alpha u_x + \frac{ju}{2(t+1)}\right)_x (0,t), \\ &= \frac{1}{\epsilon}\left(u_{tx} + u^\alpha u_{xx} + \alpha u^{\alpha-1}u_x^2 + \frac{ju_x}{2(t+1)}\right)(0,t), \\ &= \frac{1}{\epsilon}\left(V_0' + a^\alpha u_{x,2}(0,t) + \alpha a^{\alpha-1}V_0^2 + \frac{jV_0}{2(t+1)}\right). \end{aligned}$$

Similarly, we can compute the other coefficients in (5) :

$$\begin{aligned} u_{x,3}(1,t) &= \frac{1}{\epsilon}\left(V_1' + b^\alpha u_{x,2}(1,t) + \alpha b^{\alpha-1}V_1^2 + \frac{jV_1}{2(t+1)}\right), \\ u_{x,4}(0,t) &= \frac{1}{\epsilon}\left(\frac{a^\alpha}{\epsilon}V_0' - \frac{ja}{2\epsilon(t+1)^2} + a^\alpha u_{x,3}(0,t)\right. \\ &\quad \left. + \left(3\alpha a^{\alpha-1}V_0 + \frac{j}{2(t+1)}\right)u_{x,2}(0,t) + \alpha(\alpha-1)a^{\alpha-2}V_0^3\right), \\ u_{x,4}(1,t) &= \frac{1}{\epsilon}\left(\frac{b^\alpha}{\epsilon}V_1' - \frac{jb}{2\epsilon(t+1)^2} + b^\alpha u_{x,3}(1,t)\right. \\ &\quad \left. + \left(3\alpha b^{\alpha-1}V_1 + \frac{j}{2(t+1)}\right)u_{x,2}(1,t) + \alpha(\alpha-1)b^{\alpha-2}V_1^3\right). \end{aligned}$$

We require that initial and boundary conditions be compatible at $x = 0$ and $x = 1$, that is,

$$u_{x,r}(0,0) = u_0^{(r)}(0), \;\; u_{x,r}(1,0) = u_0^{(r)}(1), \;\; r = 0,1,2,3,4. \tag{7}$$

Here $u_0^{(r)}(x)$ is the r^{th} derivative of $u_0(x)$ and $u_{x,r}(0,0)$ and $u_{x,r}(1,0)$ are determined above. This results in the following conditions on $V_0(t)$, $V_1(t)$ and their first derivatives at $t = 0$:

$$\begin{aligned} &V_0(0) = u_{x,1}(0,0) = u_0'(0), \; V_1(0) = u_{x,1}(1,0) = u_0'(1), \\ &V_0'(0) = \epsilon u_{x,3}(0,0) - a^\alpha u_{x,2}(0,0) - \alpha a^{\alpha-1}V_0^2(0) - \frac{jV_0(0)}{2}, \\ &V_1'(0) = \epsilon u_{x,3}(1,0) - b^\alpha u_{x,2}(1,0) - \alpha b^{\alpha-1}V_1^2(0) - \frac{jV_1(0)}{2}. \end{aligned} \tag{8}$$

Thus, having determined $p_n(x,t)$ ($n = 2,3,4$) in terms of the unknown functions $V_0(t)$, $V_1(t)$ and their derivatives, we replace u and its first partial derivatives in the right hand side of (3) by p_n and its corresponding first partial derivatives. Then, letting $x \to 0+$ and $x \to 1-$ in the resulting equation, we obtain a system of ordinary differential equations for V_0 and V_1 as

$$\begin{aligned} V_0 &= b - a + \frac{a^{\alpha+1}}{\epsilon(\alpha+1)} + \frac{1}{\epsilon}\int_0^1 \left[(s-1)\left(\frac{\partial p_n}{\partial t} + \frac{jp_n}{2(t+1)}\right) - \frac{p_n^{\alpha+1}}{\alpha+1}\right] ds, \\ V_1 &= b - a + \frac{b^{\alpha+1}}{\epsilon(\alpha+1)} + \frac{1}{\epsilon}\int_0^1 \left[s\left(\frac{\partial p_n}{\partial t} + \frac{jp_n}{2(t+1)}\right) - \frac{p_n^{\alpha+1}}{\alpha+1}\right] ds. \end{aligned} \tag{9}$$

For $n = 2$, p_n involves the values of $u_{x,r}(x,t)$ for $r = 0,1,2$ at $x = 0$ and $x = 1$. Substituting $p_2(x,t)$ in (9) and simplifying we arrive at a system of two first order nonlinear ordinary differential equations for the unknown functions V_0 and V_1. To solve this system of ODEs, we need the initial values $V_0(0)$ and $V_1(0)$ given in (8). For a set of parameter values, $\alpha = 1$, $j = 1$, $\epsilon = 0.1$ and $a = b = 0$, we give below the system of ODEs

$$\begin{aligned} V_0'(t) &= -\left\{2.6 + \frac{0.5}{1+t}\right\} V_0(t) - 1.6V_1(t) + 0.40303\, V_0(t)V_1(t) \\ &\quad -0.315152\,(V_0^2(t) + V_1^2(t)) \qquad (10) \\ V_1'(t) &= -1.6V_0(t) - \left\{2.6 + \frac{0.5}{1+t}\right\} V_1(t) + 0.40303\, V_0(t)V_1(t) \\ &\quad -0.315152\,(V_0^2(t) + V_1^2(t)). \qquad (11) \end{aligned}$$

Similarly, when we substitute p_3 or p_4 in (9), we arrive at a system of two second order nonlinear ODEs for V_0 and V_1. Thus, we need to solve a system of ODEs for V_0 and V_1 subject to initial conditions given in (8). This is done numerically. The system of nonlinear ODEs coming from (9) are of the form

$$\frac{d^2V}{dt^2} = F(t, V, V')$$

where $V = (V_0, V_1)$ and the components of F are polynomials in V_0, V_1 for $n = 3, 4$. So the local existence and uniqueness of solution is guaranteed subject to the initial data (8). It is worthwhile to note that the Hermite interpolant solution $p_n(x,t)$ approximately solves the IBVP (1)–(3) subject to the initial profile $p_n(x,0)$ for all time t.

For our computations, we have chosen $u_0(x)$ to be $\sin(\pi x)$ such that $a = b = 0$ (compatibility condition). Figure 1 shows an excellent agreement between the simulated initial profile $p_4(x,0)$ with $\alpha = 1$, $j = 1$ and $\epsilon = 0.1$ and the initial profile $u_0(x) = \sin \pi x$; the error is of order $O(10^{-5})$. In fact, when $\alpha = 1$, $p_4(x,0)$ does not depend on j and ϵ and is given by

$$p_4(x,0) = \pi\{Q_1^4(x) + Q_1^4(1-x)\} - \pi^3\{Q_3^4(x) + Q_3^4(1-x)\}, \qquad (12)$$

the 4^{th} order Hermite interpolant of $\sin \pi x$ on $[0,1]$.

We have compared the numerical solution of the IBVP (1)–(3), obtained by a finite difference scheme due to Dawson [1], and the Hermite interpolants $p_2(x,t)$, $p_3(x,t)$ and $p_4(x,t)$ at different times for different values of α, j and ϵ. Figures 2 and 3 show the numerical and Hermite interpolant solutions $p_2(x,t)$, $p_3(x,t)$, $p_4(x,t)$ at times $t = 1, 5$ for $\alpha = 1$, $j = 1$ and $\epsilon = 0.1$.

At time $t = 1$, the maximum absolute error in $p_4(x,t)$ with respect to the numerical solution is of order $O(10^{-3})$, whereas at $t = 5$ it is of order $O(10^{-5})$. The maximum values of numerical solution of (1)–(3) for $\alpha = 1$, $j = 1$ and $\epsilon = 0.1$ at $t = 1, 5$ are 0.2325 and 0.0025 respectively. Further, the maximum absolute errors in p_2 and p_3, depicted in Figures 2 and 3, with respect to the numerical solution at $t = 1$ and $t = 5$ are $O(10^{-2})$ and $O(10^{-4})$ respectively.

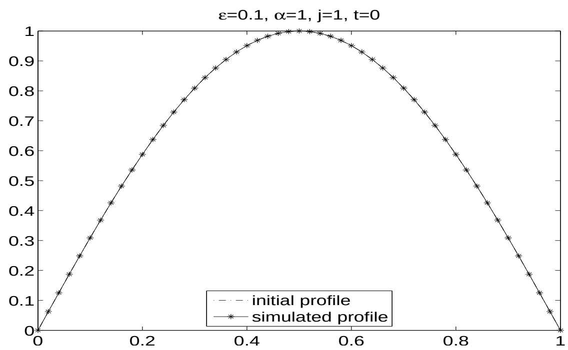

Figure 1: Comparison of initial profile $u_0(x) = \sin \pi x$ and the simulated initial profile $p_4(x,0)$ for $\epsilon = 0.1$, $\alpha = 1$ and $j = 1$.

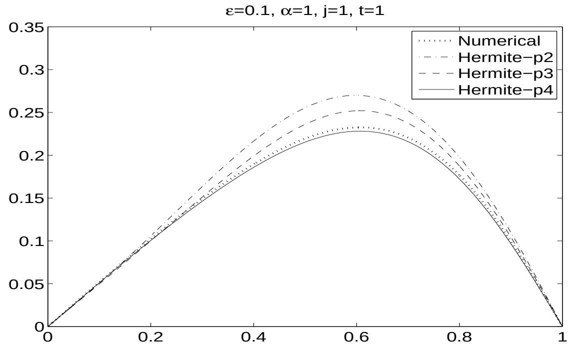

Figure 2: Comparison of numerical and Hermite interpolant solutions p_2, p_3 and p_4 for $\epsilon = 0.1$, $\alpha = 1$ and $j = 1$ at time $t = 1$.

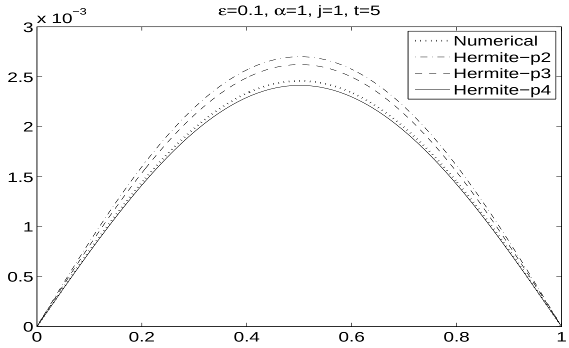

Figure 3: Comparison of numerical and Hermite interpolant solutions p_2, p_3 and p_4 for $\epsilon = 0.1$, $\alpha = 1$ and $j = 1$ at time $t = 5$.

For $\alpha = 2$, it is worthwhile to note that the compatibility conditions, as laid down in (7), are not satisfied by $u_{x,4}(0, 0)$ and $u_{x,4}(1, 0)$ with $u_0(x) = \sin \pi x$. Therefore, we compute $p_3(x, t)$ so that, $\sin \pi x$ is compatible with $p_3(x, 0)$ and is given by

$$p_3(x, 0) = \pi\{Q_1^3(x) + Q_1^3(1 - x)\} - \pi^3\{Q_3^3(x) + Q_3^3(1 - x)\}, \quad (13)$$

the 3^{rd} order Hermite interpolant of $\sin \pi x$. We have observed order of errors $O(10^{-3})$ and $O(10^{-5})$ in $p_3(x, t)$ at $t = 1, 5$, with $\alpha = 2$, $j = 1, 2$, $\epsilon = 0.1$. Further, we have verified the Hermite interpolants $p_3(x, t)$ with the finite difference numerical solutions for $\alpha = 2$, $j = 1, 2$ and $\epsilon = 0.05$. In this case also an excellent agreement is observed between the Hermite interpolants $p_3(x, t)$ and the corresponding numerical solution.

We may point out that the Hermite interpolation approximation of the solution of IBVP (1)–(3) can be used for non zero a, b also. The only requirement is the compatibility of initial and boundary data with the given PDE (see (7) and (8)). For $\alpha = 2$,

$$u_{x,4}(0, 0) \neq u_0^{(4)}(0), \ \ u_{x,4}(1, 0) \neq u_0^{(4)}(1)$$

when $u_0(x) = \sin(\pi x)$ and $a = b = 0$. That is, the compatibility condition is not satisfied. Therefore, we have not computed $p_4(x, t)$.

3 Conclusions

Inspired by Grundy's (see [3]) idea of using Hermite interpolants to approximate solutions of initial boundary value problems for nonlinear partial differential equations, we have approximated the solution of IBVP (1)–(3) by Hermite interpolants $p_n(x, t)$. In the process we arrived at a system of nonlinear ODEs (9) involving V_0 and V_1, the unknown fluxes at $x = 0$ and $x = 1$. For $n = 2$, the system of ODEs is explicitly given by (10)–(11) for a specific set of parameter values. The system of ODEs resulting from (9) for different values of n is numerically solved for V_0 and V_1 subject to initial conditions (8). Then $p_n(x, t)$ is computed at different times and compared with a numerical solution obtained by a finite difference scheme due to Dawson [1]. We have used Hermite interpolants $p_n(x, t)$ of order up to $n = 4$ because of computational simplicity and also because of the fact that $p_4(x, t)$ has agreed with the numerical solution reasonably well.

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