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Bernstein Ritz-Galerkin Method for Solving the Damped Generalized Regularized Long-Wave (DGRLW) Equation

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Abstract: In this paper a numerical method is proposed to approximate the solution of the nonlinear damped generalized regularized long-wave (DGRLW) equation with a variable coefficient. The method is based upon Bernstein Ritz-Galerkin approximations. The properties of Bernstein polynomials are first presented. These properties together with Ritz-Galerkin method are then utilized to reduce the nonlinear DGRLW equation to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: Bernstein polynomial; DGRLW equation; Ritz method; Galerkin method

1 Introduction

The damped generalized regularized long-wave (DGRLW) equation is a partial differential equation that describes the amplitude of the long-wave, which can be written as

$$u_t - (\phi(x,t)u_{xt})_x - \alpha u_{xx} + u_x + u^p u_x = f(x,t), \quad x \in \Omega, \quad 0 < t \le T, \tag{1}$$

with initial condition

$$u(x,0) = f_0(x), \quad x \in \overline{\Omega}, \tag{2}$$

and boundary conditions

$$u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad 0 \le t \le T, \tag{3}$$

where $\alpha > 0$,$p \ge 1$ is an integer, $\phi(x,t), f(x,t)$ are known functions and unknown function $u(x,t)$ being the amplitude of the long-wave at the position x and at time t.

For the mathematical theory and physical significance of the Eq. (1), we refer the reader to [1-7] and the references therein. Numerical and theoretical methods have been proposed by several researchers, based on either finite differences , finite elements ,Galerkin or Adomian decomposition scheme [8-21].

In this paper we present a new method to the solution of DGRLW equation. The method is based upon Bernstein Ritz-Galerkin approximations.

The article is organized as follows: In Section 2, we describe the basic formulation of Bernstein polynomials required for our subsequent development. Section 3 is devoted to the function approximation by using Bernstein polynomials basis and the upper bound of approximation error is presented. The properties of Ritz-Galerkin method are presented in Section 4. Section 5 is devoted to the solution of DGRLW equation by using Bernstein Ritz-Galerkin approximations. In Section 6, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples. Section 7 consists of our brief conclusions.

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2 Properties of Bernstein polynomials

The Bernstein polynomials of mth-degree are defined on the interval $[a, b]$ as [22]

$$B_{i,m}(x) = \binom{m}{i} \frac{(x-a)^i (b-x)^{m-i}}{(b-a)^m}, \quad 0 \le i \le m$$

where

$$\binom{m}{i} = \frac{m!}{i!(m-i)!}.$$

Bernstein polynomials defined above form a complete basis over the interval [a,b]. There are m + 1 mth-degree polynomials. For convenience, we set $B_{i,m}(x) = 0$, if $i < 0$ or $i > m$. A recursive definition also can be used to generate the Bernstein polynomials over $[a, b]$ so that the ith mth-degree Bernstein polynomials can be written

$$B_{i,m}(x) = \frac{(b-x)}{b-a} B_{i,m-1}(x) + \frac{x}{b-a} B_{i-1,m-1}(x).$$

It can easily be shown that each of the Bernstein polynomials is positive and also the sum of all the Bernstein polynomials is unity for all real $x \in [a, b]$, i.e., $\sum_{i=0}^{m} B_{i,m}(x) = 1$. It is easy to show that any given polynomial of degree m can be expanded in terms of linear combination of the basis functions. Some useful properties of Bernstein polynomials can be found in [24].

3 Approximation of functions

Suppose that $H = L^2[t_0, t_f]$ where $t_0, t_f \in \mathbb{R}$ and $\{B_{0,m}, B_{1,m}, \cdots, B_{m,m}\} \subset H$ be the set of Bernstein polynomials of mth-degree and

$$Y = Span\{B_{0,m}, B_{1,m}, \cdots, B_{m,m}\}$$

and f be an arbitrary element in H. Since Y is a finite dimensional vector space, f has the unique best approximation out of Y such as $y_0 \in Y$, that is

$$\exists! y_0 \in Y; \; \forall y \in Y \; ||f - y_0|| \le ||f - y||.$$

Since $y_0 \in Y$, there exist the unique coefficients $c_0, c_1, \cdots, c_m$ such that

$$f \simeq y_0 = \sum_{i=0}^{m} c_i B_{i,m} = c^T \phi, \tag{4}$$

where $\phi^T = [B_{0,m}, B_{1,m}, \cdots, B_{m,m}]$ and $c^T = [c_0, c_1, \cdots, c_m]$, and c^T can be obtained by

$$c^T < \phi \, , \, \phi > = < f, \, \phi >,$$

where

$$< f, \, \phi > = \int_{t_0}^{t_f} f(x)\phi(x)^T dx = [< f, B_{0,m} >, < f, B_{1,m} >, \cdots, < f, B_{m,m} >]$$

and $< \phi \, , \, \phi >$ is a $(m+1) \times (m+1)$ matrix and is said dual matrix of ϕ. Let

$$Q = [Q_{(i+1),(j+1)}] = < \phi \, , \, \phi > = \int_{t_0}^{t_f} \phi(x)\phi(x)^T dx,$$

then

$$c^T = \left(\int_{t_0}^{t_f} f(x)\phi(x)^T dx \right) Q^{-1}. \tag{5}$$

Theorem 1 *Suppose that H be a Hilbert space and Y be a closed subspace of H such that $dim Y < \infty$ and $\{y_1, y_2, \cdots, y_n\}$ is any basis for Y. Let x be an arbitrary element in H and y_0 be the unique best approximation to x out of Y. Then*

$$||x - y_0||^2 = \frac{G(x, y_1, y_2, \cdots, y_n)}{G(y_1, y_2, \cdots, y_n)},$$

where,

$$G(x,y_1,y_2,\cdots,y_n)=\begin{vmatrix} <x,x> & <x,y_1> & \cdots & <x,y_n> \\ <y_1,x> & <y_1,y_1> & \cdots & <y_1,y_n> \\ \vdots & \vdots & & \vdots \\ <y_n,x> & <y_n,y_1> & \cdots & <y_n,y_n> \end{vmatrix}.$$

Proof. [24] ■

The exact value of approximation error is presented by above theorem, but in the following lemma we present an upper bound to estimate the error.

Lemma 2 *Suppose that the function* $g:[t_0,t_f]\to\mathbb{R}$ *is* $m+1$ *times continuously differentiable,* $g\in C^{m+1}[t_0,t_f]$, *and* $Y=Span\{B_{0,m},B_{1,m},\cdots,B_{m,m}\}$. *If* $c^T\phi$ *be the best approximation* g *out of* Y *then the mean error bounded is presented as follows:*

$$||g-c^T\phi||\leq\frac{M(t_f-t_0)^{\frac{2m+3}{2}}}{(m+1)!\sqrt{2m+3}},$$

where $M=\max_{x\in[t_0,t_f]}|g^{(m+1)}(x)|$.

Proof. We consider the Taylor polynomial

$$y_1(x)=g(t_0)+g'(t_0)(x-t_0)+\cdots+g^{(m)}(t_0)\frac{(x-t_0)^m}{m!},$$

which we know

$$|g(x)-y_1(x)|\leq|g^{(m+1)}(\eta)|\frac{(x-t_0)^{m+1}}{(m+1)!} \quad (6)$$

where $\eta\in(t_0,t_f)$. Since $c^T\phi$ is the best approximation g out of Y, $y_1\in Y$ and using (6) we have

$$||g-c^T\phi||^2\leq||g-y_1||^2=\int_{t_0}^{t_f}|g(x)-y_1(x)|^2dx\leq\int_{t_0}^{t_f}\left[g^{(m+1)}(\eta)\frac{(x-t_0)^{m+1}}{(m+1)!}\right]^2dx$$

$$\leq\frac{M^2}{(m+1)!^{\,2}}\int_{t_0}^{t_f}(x-t_0)^{2m+2}dx=\frac{M^2(t_f-t_0)^{2m+3}}{[(m+1)!^{\,2}](2m+3)},$$

and by taking square roots we have the above bound. ■

4 The Ritz-Galerkin method

Consider the differential equation

$$L[y(x)]+f(x)=0, \quad (7)$$

over the interval $a\leq x\leq b$. Multiplying (7) by any arbitrary weight function $w(x)$ and integrating over the interval [a,b] one obtains

$$\int_a^b w(x)(L[y(x)]+f(x))dx=0, \quad (8)$$

for any arbitrary $w(x)$. Equations (7) and (8) are equivalent, because $w(x)$ is any arbitrary function.

We introduce a trial solution $u(x)$ to (7) of the form

$$u(x)=\varphi_0(x)+\sum_{j=1}^{n}c_j\varphi_j(x). \quad (9)$$

and replace $y(x)$ with $u(x)$ on the left side of (7). The residual is defined as follows

$$r(x)=L[u(x)]+f(x).$$

The goal is to construct $u(x)$ so that the integral of the residual will be zero for some choices of weight functions. That is, $u(x)$ will partially satisfy (8) in the sense that

$$\int_a^b w(x)(L[u(x)] + f(x))dx = 0.$$

for some choices of $w(x)$. One of the most important weighted residual methods was introduced by the Russian mathematician, Boris Grigoryevich Galerkin (February 20, 1871 - July 12, 1945). Galerkin's method selects the weight functions in a special way: they are chosen from the basis functions, i.e. $w(x) \in \{\varphi_i(x)\}_{i=1}^n$. It is required that the following n equations hold true

$$\int_a^b \varphi_i(x)(L[u(x)] + f(x))dx = 0 \quad i = 1, 2, ..., n.$$

To apply the method, we solve these n equations for the coefficients $\{c_j\}_{j=1}^{\infty}$.
Suppose we wish to solve a boundary value problem over the interval [a,b] with the above method, we select $\varphi_i(x), \quad i = 1, 2, ..., m$ so that satisfy the homogeneous form of the specified essential boundary conditions and φ_0 must satisfy the specified essential boundary conditions.

5 Bernstein Ritz-Galerkin method for DGRLW

Consider the DGRLW equation:

$$u_t - (\phi(x,t)u_{xt})_x - u_{xx} + u_x + u^p u_x = f(x,t), \quad 0 \le x \le 1, \quad 0 < t \le T, \tag{10}$$

With initial conditions

$$u(x,0) = f_0(x), \quad 0 \le x \le 1, \tag{11}$$

and boundary condition

$$u(0,t) = g_0(t), \quad 0 < t \le T, \tag{12}$$

$$u(1,t) = g_1(t), \quad 0 < t \le T, \tag{13}$$

Now let

$$F(u(x,t)) = u_t - (\phi(x,t)u_{xt})_x - u_{xx} + u_x + u^p u_x - f(x,t) = 0. \tag{14}$$

A Ritz-Galerkin approximation to (14) are constructed as follows. The approximation $\hat{u}$ is sought in the form of the truncated series

$$\hat{u}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij}x(x-1)tB_{i,n}(x)B_{j,m}(t) + \psi(x,t), \tag{15}$$

which $B_{i,n}(x)$ and $B_{j,m}(t)$ are Bernstein polynomials. Note that $\psi(x,t)$ is such that satisfies to the initial and boundary conditions (11-13), then it is easy to see that the approximation solution $\hat{u}(x,t)$ also satisfies the initial and boundary conditions (11-13). This approximation provides greater flexibility in which to impose initial and boundary conditions.

In general, $\psi(x,t)$ is not unique. We choose $\psi(x,t)$ as follows:

If $g_0(0) \neq 0$, let $\psi(x,t)$ to be interpolating function for $\psi(0,t) = g_0(t)$ and $\psi(1,t) = g_1(t)$, that is

$$\psi(x,t) = g_0(t)(1-x) + xg_1(t).$$

Now let

$$\psi(x,t) = g_0(t)k(x) + xg_1(t),$$

and we choose $k(x)$ such that $k(1) = 0, \quad k(0) = 1$, and $\psi(x,0) = f_0(x)$. we have

$$\psi(x,0) = g_0(0)k(x) + xg_1(0) = f_0(x),$$

thus

$$k(x) = \frac{f_0(x) - xg_1(0)}{g_0(0)},$$

and since $f_0(1) = g_1(0)$ and $f_0(0) = g_0(0)$, thus $k(1) = 0$, and $k(0) = 1$. Therefore

$$\psi(x,t) = g_0(t)(\frac{f_0(x) - xg_1(0)}{g_0(0)}) + xg_1(t), \tag{16}$$

satisfies to the initial and boundary conditions (11-13).

Now if $g_0(0) = 0$ and $g_1(0) \neq 0$ let

$$\psi(x,t) = g_0(t)(1-x) + k(x)g_1(t),$$

and we choose $k(x)$ such that $k(0) = 0, \quad k(1) = 0$, and $\psi(x,0) = f_0(x)$. From the initial and boundary conditions (11-13) we obtain

$$\psi(x,t) = g_0(t)(1-x) + g_1(t)\frac{f_0(x)}{g_1(0)}. \tag{17}$$

Finally let $g_0(0) = g_1(0) = 0$. In this case we choose

$$\psi(x,t) = g_0(t)(1-x) + xg_1(t) + k(x,t),$$

and we obtain $k(x,t)$ such that $k(0,t) = k(1,t) = 0$, and $\psi(x,0) = f_0(x)$. Since $g_0(0) = g_1(0) = f_0(0) = f_0(1) = 0$, thus we can choosing $k(x,t) = f_0(x)$, and therefore

$$\psi(x,t) = g_0(t)(1-x) + xg_1(t) + f_0(x), \tag{18}$$

that satisfies to the initial and boundary conditions (11-13).

Now the expansion coefficients c_{ij} are determined by the Galerkin equations

$$< F(\hat{u}), B_{i,n}(x)B_{j,m}(t) >= 0, \tag{19}$$

where $< . >$ denotes the inner product defined by

$$< F(\hat{u}), B_{i,n}B_{j,m} >= \int_0^1 \int_0^T F(\hat{u})(x,t)B_{i,n}(x)B_{j,m}(t)dtdx.$$

Equation (19) give a system of nonlinear equations which can be solved for the elements of c_{ij} using the Newton's iterative method. In next section three examples are solved using the method described.

6 Illustrative examples

6.1 Example 1

Consider the nonlinear homogenous DGRLW equation[21]

$$u_t - (\phi(x,t)u_{xt})_x - \alpha u_{xx} + u_x + u^p u_x = 0, \quad 0 < x < 1, \quad 0 < t \leq 1, \tag{20}$$

with initial condition

$$u(x,0) = exp(-x), \quad 0 \leq x \leq 1,$$

and boundary conditions

$$u(0,t) = exp(2t), \quad u(1,t) = exp(2t-1), \quad 0 \leq t \leq 1,$$

where

$$\phi(x,t) = -\frac{1}{6}exp(-2x+4t),$$

and $p = 2, \quad \alpha = 1$, whose exact solution is

$$u(x,t) = exp(-x+2t).$$

We applied the method presented in this paper with $m = n = 2$ and solved Eq.(20). From Eq. (16) we have

$$\psi(x,t) = exp(-x)exp(2t),$$

and from Eq. (19) we obtain

$$c_{ij} = 0, \quad i,j = 0,1,2.$$

Thus from (15) we have

$$\hat{u}(x,t) = exp(-x+2t),$$

which is the exact solution.

6.2 Example 2

Consider the nonlinear inhomogenous DGRLW equation[21]

$$u_t - (\phi(x,t)u_{xt})_x - \alpha u_{xx} + u_x + u^p u_x = f(x,t), \quad 0 < x < 1, \quad 0 < t \leq 1, \tag{21}$$

with initial condition

$$u(x,0) = sin(x), \quad 0 \leq x \leq 1,$$

and boundary conditions

$$u(0,t) = 0, \quad u(1,t) = sin(1)exp(-t), \quad 0 \leq t \leq 1,$$

where

$$\phi(x,t) = xt,$$

and

$$f(x,t) = (tcos(x) - xtsin(x) + cos(x) + sin(x)cos(x)exp(-t))exp(-t),$$

and $p = \alpha = 1$ whose exact solution is

$$u(x,t) = sin(x)exp(-t).$$

We applied the method presented in this paper with $m = n = 2$ and solved Eq.(21). From Eq. (17) we have

$$\psi(x,t) = sin(x)exp(-t),$$

and from Eq. (19) we obtain

$$c_{ij} = 0, \quad i,j = 0,1,2.$$

Thus from (15) we have

$$\hat{u}(x,t) = sin(x)exp(-t),$$

which is the exact solution.

6.3 Example 3

Consider the nonlinear inhomogenous DGRLW equation[13]

$$u_t - (\phi(x,t)u_{xt})_x - \alpha u_{xx} + u_x + u^p u_x = f(x,t), \quad 0 < x < 1, \quad 0 < t \leq 1, \tag{22}$$

with initial condition

$$u(x,0) = x(x - \frac{1}{2})(x-1), \quad 0 \leq x \leq 1,$$

and boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 \leq t \leq 1,$$

where

$$\phi(x,t) = (x^2+1)exp(-\frac{t}{10}),$$

and

$$f(x,t) = \frac{1}{20}exp(-\frac{t}{10})(-2x^3 + 63x^2 - 181x + 70 + (60x^5 - 150x^4 + 15x^3 - 63x^2 + 19x - 6)exp(-\frac{t}{10})),$$

and $p = \alpha = 1$ whose exact solution is

$$u(x,t) = x(x - \frac{1}{2})(x-1)exp(-\frac{t}{10}).$$

We applied the method presented in this paper with $m = n = 3$ and solved Eq.(22). From Eq. (18) we have

$$\psi(x,t) = x(x - \frac{1}{2})(x-1).$$

The exact and approximate solution are plotted in figure 1.

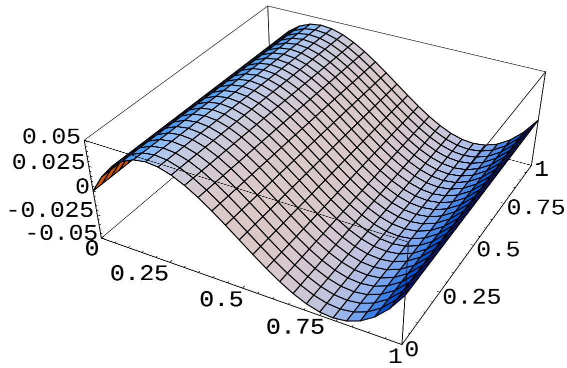

Figure 1: Exact and Approximate Solution of $u(x,t)$.

7 Conclusion

The properties of the Bernstein polynomials together with the Ritz-Galerkin method are used to reduce the solution of the DGRLW Equation to the solution of algebraic equations. The choice of basis and $\psi(x,t)$, provides greater flexibility in which to impose initial and boundary conditions. Moreover, only a small number of Bernstein polynomials basis are needed to obtain a satisfactory result. Illustrative examples are included to demonstrate the validity and applicability of the new technique. It is also shown that the Bernstein Ritz-Galerkin method provides an exact solution for the DGRLW problem. The given numerical examples support this claim.

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