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Analytical Solutions of Unsteady Oscillatory Particulate Visco-Elastic Fluid Between Two Parallel Walls

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Abstract: Closed-form solutions for the velocity and temperature profiles have been obtained for a particulate visco-elastic fluid between two oscillatory walls in the presence of transverse magnetic field. The flow is made possible by the continous actions of the oscillatory walls and the time dependent pressure gradient. Parametric analysis involving some parameters embedded in the system have been carried out to elucidate some interesting features in the system. The shear stress and the Nusselt numbers at the wall have also been obtained and demonstrated graphically.

Keywords: Closed-form; visco-elastic; particulate-fluid system; oscillatory wall; Nusselt numbers; shear stress

1 Introduction

The phenomenon of heat and mass transfer of a particle-fluid system in parallel walls has continued to attract the attentions of researchers in the last few decades. This may not be unconnected with its wide range of applications to a number of scientific and technological processes such as fluidization, combustion, petroleum industry, the use of dust in gas cooling system, fluid droplets spray, polymer technology, blood transport in the arteries and the purification of drinking water. The pioneering work of saffman [1] on the fluid-particle system has continue to spur reasearch through which the frontier of knowledge in the subject area has recorded significant contributions.The majority of contributions in the area have been in the area of isothermal incompressible particulate-viscoelastic flow[3,7,9-11], while others have been in the non-isothermal particulate viscoelastic flow between two infinite walls at various thermal conditions[2,4-6,8,12-13]. Inspite of the numerous works in literature, the study of non-isothermal particulate flow has received less attentions and to the best of our knowledge, the use of varying pressure gradient has often made analytical solution impossible.

Motivated by this, the current paper is aimed at investigating the influence of some parameters embedded in the system on the behaviour of the velocity and temperature fields using an analytical framework. In each case, we calculate the shear stress, skin friction at the boundary walls and further demonstrate these solutions graphically.

2 Basic Equations and Problem Formulation

In cartesian coordinate system, we consider a two dimensional unsteady, incompressible, plane viscous fluid between two parallel oscillatory plates whose equations are $y = -h$ and $y = h$(distance $2h$ apart). Let x-axis be along the flow of liquid at the fixed wall and y-axis perpendicular to it. A uniform magnetic field of strength B_0 is applied perpendicular to the flow region. In addition, other physically reasonable assumptions are as follows: (i) the solid particles are spherical solid, non-conducting, equal size and uniformly distributed in the flow; (ii) the interactions between the particles, chemical reactions between the solid and fluid are neglected respectively; (iii) the particulate system has equal and constant density; (iv) and the wall temperature condition oscillates in time with frequency w. Thus, the dimensional conservation relations for the mass, momentum and energy are

$$\frac{\partial u}{\partial x} = 0, \tag{1}$$

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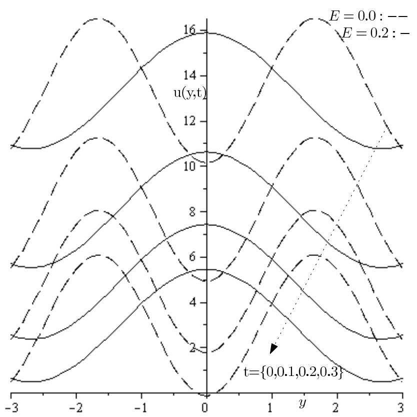

Figure 1: Plot of u vs y for $w_1 = \pi$,$\lambda = 1$, $E = \{0, 0.2\}$,$G = 10$ and $q = 5$.

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \beta\frac{\partial^3 u}{\partial y^2 \partial t} - \frac{\sigma B_0^2}{\rho}u - \frac{kN}{\rho}(u - v), \tag{2}$$

$$\frac{\partial v}{\partial x} = 0, \tag{3}$$

$$\frac{\partial v}{\partial t} = \frac{k}{m}(u - v), \tag{4}$$

$$\frac{\partial T}{\partial x} = 0, \tag{5}$$

$$\frac{\partial T_f}{\partial t} = \frac{\partial}{\partial y}\left(K(T_f)\frac{\partial T_f}{\partial y}\right) + \frac{\lambda'(T_f)}{C\rho}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma B_0^2}{\rho}u^2 + \frac{1}{\gamma_T}(T_p - T_f), \tag{6}$$

$$\frac{\partial T_p}{\partial t} = -\frac{1}{\gamma_T}(T_p - T_f). \tag{7}$$

The relevant boundary conditions are

$$t \le 0, \quad u = v = 0, \tag{8a}$$

$$t > 0, \quad y = -h, \quad u = U(t), \tag{8b}$$

$$y = h, \quad u = V(t), \tag{8c}$$

$$t \le 0, \quad T_f = T_p = T_1, \tag{9a}$$

$$t > 0, \ y = -h, \ T_f = T_p = T_1, \tag{9b}$$

$$y = h, \quad T_f = T_p = T_2, \tag{9c}$$

$$T_w(t) = \bar{T_w} + \epsilon\left(\bar{T}_w - T_1\right)e^{-iwt}, \tag{9d}$$

where u and v denote the local velocity vectors of fluid and dust particles respectively, T_f and T_p are temperature of the fluid and dust particles respectively, while x and y are the spatial variables in the direction and perpendicular to the flow respectively and t is the time variable. ρ is the density, P is the static fluid pressure, ν is the kinetic viscosity, N is the number of dust particles per unit volume and k is a resistance coefficient, K is the thermal conductivity, γ_T is the temperature relaxation time, B_0 is a constant magnetic field parameter, σ is the electrical conductivity, m is the mass per unit volume of the particle, T_1 is the quiscent ambient fluid temperature, $\bar{T_w}$ is the average wall temperature, $T_w(t)$ is the oscillating temperature of the wall, λ' is the fluid viscosity and μ is the fluid diffusivity coefficient, β is the coefficient of visco-elasticity and w is the wall temperature oscillation amplitude.

The governing equations (1)-(9) can be simplified by writing in the non-dimensional forms using the variables

$$x' = \frac{x}{h}, t' = \frac{\nu}{h^2}t, \ y' = \frac{y}{h}, \ P' = \frac{h^2 P}{\rho\nu^2},$$

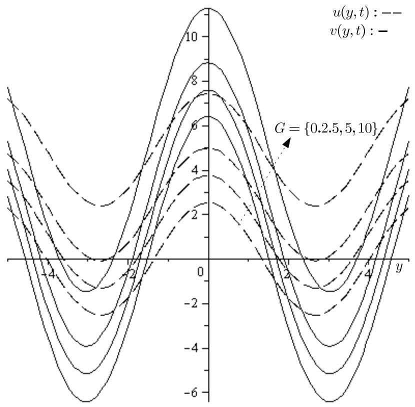

Figure 2: Plot of v vs y for $w_1 = \pi$,$\lambda = 1$, $E = \{0, 0.2\}$,$G = 10$ and $q = 5$.

$$u' = \frac{uh}{\nu},\ v' = \frac{vh}{\nu},\ T_f' = \frac{T_f - T_1}{T_2 - T_1}, T_p' = \frac{T_p - T_1}{T_2 - T_1}.$$

After dropping primes, the non-dimensional form of equations $(1) - (9)$ are

$$\frac{\partial u}{\partial x} = 0, \tag{10}$$

$$\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \frac{\partial^2 u}{\partial y^2} + E\frac{\partial^3 u}{\partial y^2 \partial t} - Hu + \lambda(v - u), \tag{11}$$

$$\frac{\partial v}{\partial x} = 0, \tag{12}$$

$$\frac{\partial v}{\partial t} = L(u - v), \tag{13}$$

$$\frac{\partial T_f}{\partial x} = 0, \tag{14}$$

$$\frac{\partial T_f}{\partial t} = \frac{1}{P_r}\frac{\partial^2 T_f}{\partial y^2} + E_c\left(\frac{\partial u}{\partial y}\right)^2 + E_c H^2 u^2 + \alpha(T_p - T_f), \tag{15}$$

$$\frac{\partial T_p}{\partial x} = 0, \tag{16}$$

$$\frac{\partial T_p}{\partial t} = -\alpha(T_p - T_f). \tag{17}$$

The relevant boundary conditions are

$$t \leq 0, \quad u = v = 0, \tag{18a}$$

$$t > 0, \quad y = -h,\ u = U(t), \tag{18b}$$

$$y = h, \quad u = V(t), \tag{18c}$$

$$t \leq 0, \quad T_f = T_p = 0, \tag{19a}$$

$$t > 0, \quad y = -h,\ T_f = T_p = 0, \tag{19b}$$

$$y = h, \quad T_f = T_p = 1, \tag{19c}$$

and an oscillatory wall temperature (see Dora and Horst [13])

$$T_w(t) = \bar{T_w} + \epsilon\left(\bar{T_w} - T_1\right)e^{-iwt}, \text{ where} \tag{19d}$$

$E = \frac{\beta}{h^2}$, visco-elastic parameter,
$L = \frac{Nh^2}{\nu} = \frac{kh^2}{m\nu}$, reciporcal of relaxation time of particle,
$\lambda = \frac{Lk}{\rho}$, mass concentration of particle,
$H^2 = \frac{\sigma B_0^2 h^2}{\nu\rho}$, square of the Hartman's number,
$E_c = \frac{\lambda'\nu}{\rho h^2 C(T_2-T_1)}$, Eckert number,
$P_r = \frac{\nu\rho C}{K}$, Prandtl number,
$\alpha = \frac{h^2}{\nu\gamma_T}$, temperature relaxation time parameter.

From the continuity equations((10), (12), (14) and (16))

$$u = u(y),\ v = v(y),\ T_f = T_f(y) \ \text{ and }\ T_p = T_p(y). \tag{20}$$

In line with the physics of the problem, we suppose

$$U(t) = e^{-iw_1 t},\ V(t) = e^{-iw_2 t} \text{ and } \frac{\partial P}{\partial x} = Ge^{-qt}, \quad q > 0, \tag{21}$$

where w_1 and w_2 are the amplitudes of the oscillatory upper and lower walls respectively.

2.1 Momentum equations

By differentiating equation (11) with respect to t,

$$u_{tt} = Gqe^{-qt} + u_{yyt} + Eu_{yytt} + \lambda(v_t - u_t) - Hu_t, \tag{22}$$

and rearrangement of (11) gives

$$u - v = \frac{-1}{\lambda}\left(u_t + Ge^{-qt} - u_{yy} - Eu_{yyt} + Hu\right). \tag{23}$$

Substituting (23) into (22), we obtain

$$\begin{aligned} u_{tt} = Eu_{yytt} + (1+LE)\,u_{yyt} + Lu_{yy} - (L+\lambda+H)\,u_t - HLu \\ - G(L-q)e^{-qt}. \end{aligned} \tag{24}$$

From the boundary conditions, we propose ansatz in the form[8]

$$u = e^{-iw_1 t} f(y) + e^{-iw_2 t} g(y) + R(t), \tag{25}$$

where $f(y)$ and $g(y)$ are functions to be determined and

$$f(-1) = 1,\ \ f(1) = 0,\ \ g(-1) = 0,\ \ g(1) = 1,\ \ R(t) \to 0,\ \ t \to \infty. \tag{26}$$

Substituting (25) into (24) and collecting terms($e^{-iw_1 t}$, $e^{-iw_2 t}$, e^0)

$$f'' + Q^2 f = 0, \tag{27}$$

$$g'' + S^2 g = 0, \tag{28}$$

$$\frac{dR}{dt} = \frac{G(L-q)}{(L+\lambda+H)}e^{-qt}, \tag{29}$$

where $Q^2 = \left(\frac{w_1^2 - HL + iw_1(L+\lambda+H)}{L - Ew_1^2 - iw_1(1+LE)}\right)$ and $S^2 = \left(\frac{w_2^2 - HL + iw_2(L+\lambda+H)}{L - Ew_2^2 - iw_2(1+LE)}\right)$.

Solving equations $(27) - (29)$, subject to the boundary condition (26)

$$f(y) = \frac{\sin(1-y)Q}{\sin 2Q}, g(y) = \frac{\sin(1+y)S}{\sin 2S} \text{ and } R(t) = \frac{G(q-L)}{q(L+\lambda+H)}e^{-qt},$$

and

$$u = \frac{\sin(1-y)Q}{\sin 2Q}e^{-iw_1 t} + \frac{\sin(1+y)S}{\sin 2S}e^{-iw_2 t} + \frac{G(q-L)}{q(L+\lambda+H)}e^{-qt}. \tag{30}$$

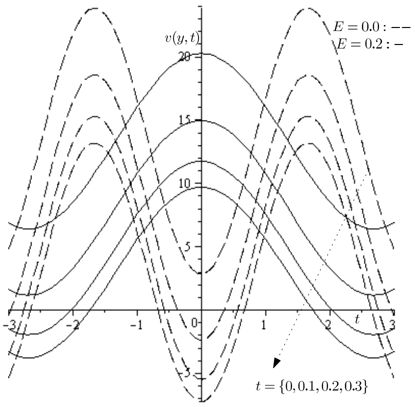

Figure 3: Plot of u, v vs y for $w_1 = \pi$,$\lambda = R = 1$,$E = 0.2$,$q = 5$ & $t = 0.2$.

Substituting (30) into (23), we obtain

$$v = M\frac{\sin(1-y)Q}{\sin 2Q}e^{-iw_1 t} + N\frac{\sin(1+y)S}{\sin 2S}e^{-iw_2 t} + G_c e^{-qt}, \tag{31}$$

$$M, N = 1 + \frac{iw_{1,2}(E-1)}{\lambda} + \frac{H-1}{\lambda},\ G_c = \frac{G}{\lambda}\left(1 + \frac{(q-L)(1+\lambda-q)}{q(L+\lambda+H)}\right).$$

2.2 Energy equations

We shall apply similar procedure to solve the energy equation

$$T_{f_{tt}} = \tfrac{1}{P_r}T_{f_{yyt}} + \tfrac{\alpha}{P_r}T_{f_{yy}} + E_c\left[\alpha u_y^2 + (u_y)_t^2 + \alpha H^2 u^2 + H^2(u)_t^2\right] - 2\alpha T_f, \tag{32}$$

by taking taking ansatz in line with (19d)

$$T_f(y,t) = \Lambda(t) + \epsilon_1\theta(y)\phi(t) + \epsilon_2\psi(y)e^{-iwt}, \tag{33}$$

where,

$$\theta(-1) = 0,\ \theta(1) = 0,\ \ \psi(-1) = 0,\ \psi(1) = 0, \Lambda(0) = 0,\ \phi(0) = 1, \tag{34}$$

ϵ_1, ϵ_2 are small amplitude and ϵ_1=$\epsilon_2 = 0$ when $t = 0$. If we put (33) into (32), assume that $w = 2w_1 = 2w_2$ and taking terms(e^0, $e^{-(q+iw_1)t}$, $e^{-2iw_1 t}$)

$$\Lambda'' + 2\alpha\Lambda' = \frac{G^2(\alpha-2q)(q-L)^2}{q^2(L+\lambda+H)^2}e^{-2qt}, \tag{35}$$

$$\theta'' - P_r\theta' = \frac{2G(\alpha-(q+iw_1))(L-q)}{q^2(L+\lambda+H)^2\cos S}\cos Sy, \tag{36}$$

$$\psi'' + a\psi + b + c\cos 2(1-y)S + d\cos(Sy) + e\cos 2(1+y)S = 0, \tag{37}$$

provided

$$\phi(t) = e^{-(q+iw_1)t},\ q + iw_1 = \frac{(2\alpha-1) \pm \sqrt{4\alpha^2 - 8\alpha + 1}}{2}, \tag{38}$$

$$\begin{aligned} &a = \tfrac{4w_1 P_r(w_1+i\alpha)}{\alpha-2iw_1},\ c = \left[\tfrac{H^2}{2\sin^2 2S} + \tfrac{S^2(\alpha-4iw_1^2)}{2(\alpha^2+4w_1^2)} + i\tfrac{4S^2\alpha w_1}{2(\alpha^2+4w_1^2)}\right], \\ &b = \tfrac{E_c P_r(H^2+S^2)}{2\sin^2 2S},\ d = \tfrac{E_c(H^2+S^2)}{2\sin^2 2S},\ e = \tfrac{E_c(S^2-H^2)}{2\sin^2 2S}. \end{aligned} \tag{39}$$

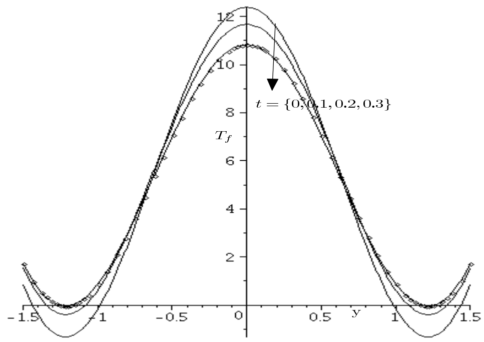

Figure 4: Plot of T_f vs y for $w_1 = \pi$, $E = 0.2$,$G = 10$ and $q = 5$.

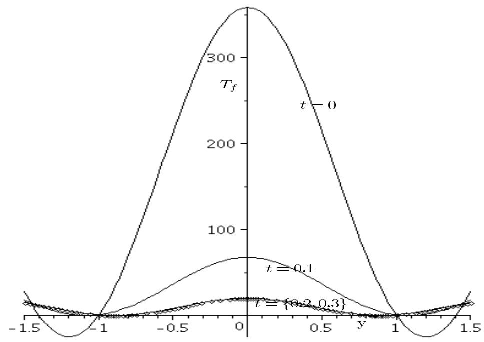

Figure 5: Plot of T_f against y for $w_1 = \pi$, $E = 0.0$,$G = 10$ and $q = 5$.

After solving equations $(35) - (37)$ subject to (34), we obtain

$$\Lambda(t) = \frac{G^2(2q-\alpha)(q-L)^2}{4q^3(\alpha-q)(L+\lambda+H)^2}\left(e^{-2qt}-1\right), \tag{40}$$

$$\theta(y) = \frac{B\epsilon_1}{\cos S}\left(2e^{2\sqrt{P_r}}\cos\left(2S\right)\cosh\sqrt{P_r}y - (1+e^{2\sqrt{P_r}})\cos\left(2Sy\right)\right), \tag{41}$$

$$\phi(t) = e^{-(q+iw_1)}, \tag{42}$$

$$\begin{gathered}\psi(y) = \tfrac{1}{2(a-4S^2)}\left[-2b + 8b/aS^2 - 2c\cos 2S(y-1) - 2d\cos(2Sy)\right. \\ -2e\cos 2S(y+1) + \left(2b - 8b/aS^2 + 2d\cos(2S) + (c+e)\sec\sqrt{a}\times\right. \\ \left.(1+\cos(4S))\right)\cos\left(\sqrt{a}y\right) + 2(c-e)\csc\sqrt{a}\sin^2\left(2S\right)\sin\left(\sqrt{a}y\right)\left.\right],\end{gathered} \tag{43}$$

$$\begin{gathered}T_f(y,t) = A\left(e^{-2qt}-1\right) + \epsilon_2\psi(y)e^{-2iw_1t} + \\ B\left[2e^{\sqrt{P_r}}\cos(2S)\cosh(\sqrt{P_r}y) - (1+e^{2\sqrt{P_r}})\cos(2Sy)\right]e^{-(q+iw_1)t}.\end{gathered} \tag{44}$$

Substituting (44) into (7) and using the boundary conditions $(9b)$-$(9c)$

$$\begin{gathered}T_p(y,t) = \left\{\tfrac{\alpha A}{\alpha-2q}e^{-2qt} - A + \right. \\ \tfrac{B\alpha}{\alpha-q-iw_1}\left[2e^{\sqrt{P_r}}\cos(2S)\cosh(\sqrt{P_r}y) - (1+e^{2\sqrt{P_r}})\cos(2Sy)\right]\times \\ \left. e^{-(q+iw_1)t} + \tfrac{\alpha\epsilon_2}{2a(a-4S^2)(\alpha-2iw_1)}\psi(y)e^{-2iw_1t}\right\},\end{gathered} \tag{45}$$

where $A = \frac{G^2(2q-\alpha)(q-L)^2}{4q^3(\alpha-q)(L+\lambda+H)^2}$, $B = \frac{2\epsilon_1 G(L-q)[\alpha-(q+iw_1)]}{q(P+4S^2)(1+e^{2\sqrt{P_r}})(L+\lambda+H)}$.

2.3 Skin friction and Nusselt number

It may be necessary to evaluate the wall shear stress(τ_f) and the rate of heat transfer(N_u) at the wall surface surface.

The skin friction τ_f at $y = \pm 1$ are given by

$$\tau_f = \frac{\partial u}{\partial y}\mid_{y=\pm 1} = 1, \tag{46}$$

while the Nusselt number is

$$\begin{gathered}N_u\mid_{y=\pm1} = \tfrac{\partial T}{\partial y}\mid_{y=\pm1} = \epsilon_2\tfrac{\partial\psi}{\partial y}\mid_{y=\pm1} e^{-2iw_1t} + \\ B\left[\pm 2\sqrt{P_r}e^{2\sqrt{P_r}}\sin(2S)\sinh(\pm\sqrt{P_r}) - (1+e^{2\sqrt{P_r}})\sin(\pm 2)\right]e^{-(q+iw_1)t}.\end{gathered} \tag{47}$$

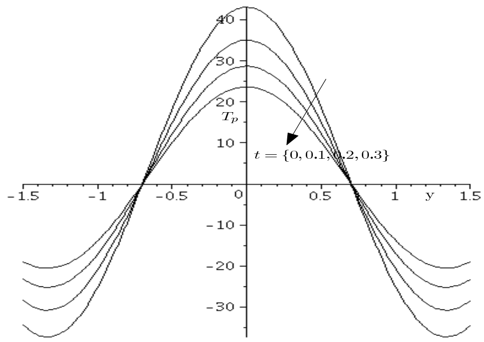

Figure 6: Plot of T_p vs y for $w_1 = \pi$, $E = 0.2$,$G = 10$, $\alpha = 2$ and $q = 5$.

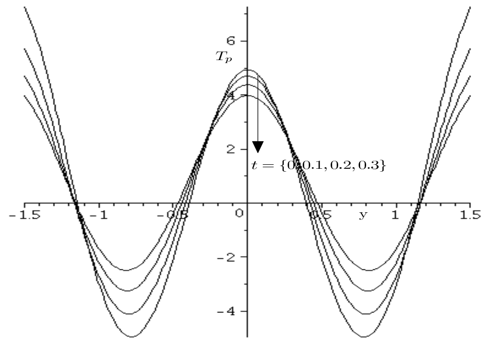

Figure 7: Plot of T_p vs y for $w_1 = \pi$,$E = 0.0$, $G = 10$,$\alpha = 2$ and $q = 5$.

3 Conclusion

The present investigation highlights the behaviour of non-isothermal particulate viscoelastic fluid subjected to a time varying pressure gradient. The systems of momentum and heat transfer equations are solved analytically and analysed using the visco-elastic(E), pressure gradient(G), Prandtl(Pr) and Eckert(Ec) parameters. Figures 1 and 2 are plots of fluid and particle velocities respectively, with an observed monotonic decrease in velocity with time. In the domain $0 \leq y \leq 1$, the viscoelastic fluid velocity decreases to the wall, while it is on the contrary for the non-viscoelastic fluid($E \neq 0$).

In figure 3, the particle velocities are always upper limits when compared with the fluid velocities. Furthermore, the velocities decrease to the wall for some values pressure gradient parameter.

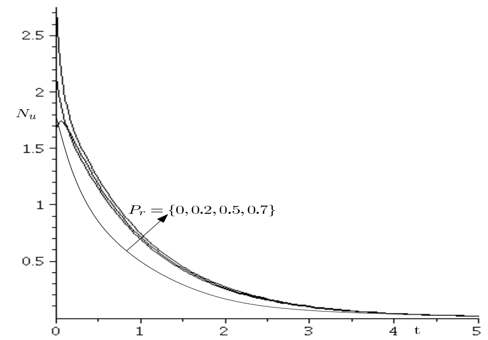

Figure 8: Plot of N_u vs t for $w_1 = \pi$, $E = 0.2$,$G = 10$ and $q = 5$.

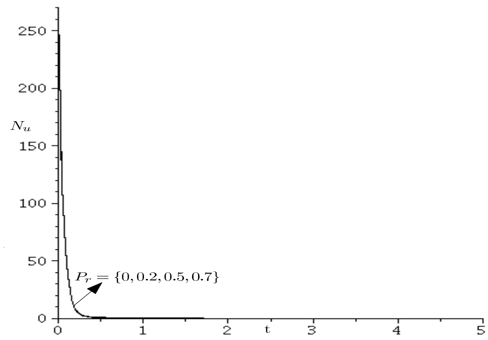

Figure 9: Plot of N_u vs t for $w_1 = \pi$, $E = 0.0$,$G = 10$ and $q = 5$.

The behaviour of fluids temperature(T_f) for $E = 0.2$ and $E = 0$ are shown in Figures 4 and 5 respectively. We observed that there is a remarkable disparity in the temperature profiles of the visco-elastic($E = 0.2$) and that of non-visco-elastic fluids($E = 0$). This behaviour may be due to the possibility of higher thermal conductivity for the viscoelastic system. Although the particle temperature(T_p) decreases with increasing time, the viscoelastic system(figures 6) has higher temperature profile as compared with the the non-viscoelastic system(figure 7). Despite the decrease in the Nusselt number(N_u) with time(t), figure 8 shows that the N_u is monotonically increasing with the Prandtl number(Pr) for the viscoelastic fluid, whereas there is no observed significant effect for the non-viscoelastic fluid(Figure 9).

We have obtained analytical solutions which have proved to be a generalization of known ones[2, 7]. Since analytic solutions are exact and self-contained, it is expected that these solutions may serve as a possible benchmark to develop and validate other forms of solutions in the theory and application of heat and mass transfer.

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