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Compacton Solutions and Peakon Solutions for a Coupled Nonlinear Wave Equation

Dianchen Lu *, Guangjuan Yang
Faculty of Science, Jiangsu University
Zhenjiang, Jiangsu, 212013, P.R.China
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Abstract: A coupled nonlinear wave equation is studied in the present paper. With the aid of Mathematica and Wu elimination method, through different ansatze, more solitary solutions, including compacton solutions, peakon solutions, as well as traveling solutions are found in this paper.

Key word: compacton solutions; coupled nonlinear wave equation; peakon solutions; solitary solutions; ansatze method

1 Introduction

The investigation of exact solutions for nonlinear evolution equations has important academic and actual value. For a long time, this work is of special interest to mathematicians and physicists. A number of methods were presented, such as inverse scatting theory, Hirota's bilinear methods, the truncated Painlere' expansion, homogeneous balance method, the sine-cosine method and other methods [1]-[3]. Recently, there are lots of results about coupled nonlinear equations. Guha-Roy [4]-[6] analyzed the following coupled nonlinear wave equation

$$\begin{cases} u_t + \alpha v^2 v_x + \beta u^2 u_x + \lambda u u_x + \gamma u_{xxx} = 0 \\ v_t + \delta (uv)_x + \varepsilon v v_x = 0 \end{cases}, \tag{1.1}$$

where $\alpha, \beta, \gamma, \lambda, \delta$ and ε are any arbitrary constants. When $v = 0$ is correct, Eq.(1.1) turns to kdv and mkdv, which is generally used in solid-state physics, plasma physics, hydro physics, quanta field theory and so on. Guha-Ray [4]-[5] suppose $|\xi| = |x - ct| \to +\infty$ with $u(\xi), u'(\xi), u''(\xi) \to 0$, by transformation $u(\xi) = \frac{1}{\phi(\xi)}$ and Weierstrass ellipse function, some exact solitary wave solutions and Cnoidal solution in special conditions are obtained. Shang [7], some new explicit and exact traveling wave solutions to the coupled nonlinear wave equation are present through two different ansatze. Those solutions include the bell-shaped solitary wave solutions which have non-zero asymptotic value, the kink-shaped and antikink-shaped solitary wave solutions, singular traveling wave solutions and periodic wave solutions of the triangular function type. Xu[8], new exact and explicit solitary solutions are obtained for the coupled nonlinear wave equation by using a simple way of making the ansatze $u(x,t) = A \sec h^n(\xi) + A_0$ with $\xi = k(x + \omega t) + \xi_0$. Compacton solutions are zero outside a finite domain of space variable x. Peakon solutions have discontinuous first derivative on the peak. To those kinds of solitary wave solutions of the above coupled nonlinear wave equation, fewer results have been carried out. The main aim of the paper is to present such special solitary wave solutions. Our method is a direct way similar as Tian [9]-[11]. In the past few years, function-series method [9]-[11] has been systematized to obtain solitary solutions of nonlinear equations. Functions in the method were chosen to be hyperbolic secant, sine, cosine, etc. We use this method to give more new solitary solutions for the coupled nonlinear wave equations.

* Corresponding author: E-mail: dclu@ujs.edu.cn

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This paper is organized in five sections. In section 2, we study compacton solutions of the coupled nonlinear wave equation by direct sine and cosine method. We find peakon solutions in section 3. Some solitary solutions are obtained in section 4. The conclusion is given in section 5.

2 Compacton solutions

First we assume that

$$v = Au + B, \tag{2.1}$$

where A and B are constants to be determined later. We seek compacton solutions for Eq.(1.1), that is solution whose energy is limited in a finite domain.

We set

$$u(x,t) = u(\xi), v(x,t) = v(\xi), \xi = k(x - Dt) \quad , \tag{2.2}$$

where k denotes wave numbers and D is velocity. Substituting (2.1) and (2.2) into Eq.(1.1), Eq.(1.1) is now

$$\begin{cases} (-D + \alpha B^2 A)u' + (2A^2\alpha B + \lambda)uu' + (\alpha A^3 + \beta)u^2u' + \gamma k^2 u''' = 0 \\ (-DA + \delta B + \varepsilon BA)u' + uu'(2\delta A + \varepsilon A^2) = 0 \end{cases} , \tag{2.3}$$

where $' \equiv \frac{d}{d\xi}$.

In order to find compacton solutions, we suppose that Eq.(1.1) has the following traveling wave form solutions:

$$u(\xi) = \begin{cases} R\cos^m(\xi), & |\xi| \le \frac{\pi}{2} \\ 0, & |\xi| > \frac{\pi}{2} \end{cases} , \tag{2.4}$$

and

$$u(\xi) = \begin{cases} R\sin^m(\xi), & |\xi| \le \frac{\pi}{2} \\ 0, & |\xi| > \frac{\pi}{2} \end{cases} , \tag{2.5}$$

where R and m are constants to be determined .

Substituting (2.4) into system (2.3) and collecting all terms with the same power in $\cos(\xi)$, we get

$$\begin{array}{l} -Rm(-D + \alpha B^2 A - k^2m^2\gamma)\cos^{m-1}(\xi) + (\alpha A^3 + \beta)R^3 m \cos^{3m-1}(\xi) \\ +(2A^2\alpha B + \lambda)R^2 m \cos^{2m-1}(\xi) - \gamma k^2 Rm(m-1)(m-2)\cos^{m-3}(\xi) = 0 \end{array} , \tag{2.6}$$

$$(-AD + \delta B + \varepsilon AB)Rm\cos^{m-1}(\xi) + R^2 m\cos^{2m-1}(\xi)(2\delta A + \varepsilon A^2) = 0 \quad . \tag{2.7}$$

Letting all coefficients of $\cos(\xi)$ in Eq.(2.6) and (2.7) to zero we can get a set of algebraic polynomials with respect to the unknown variables $A, B, D, m, R, k, \delta, \alpha, \beta, \varepsilon$.

$$\begin{cases} m^2 - 3m + 2 = 0 \\ D - \alpha B^2 A + k^2 m^2 \gamma = 0 \\ 2A^2 B\alpha + \lambda = 0 \\ \alpha A^3 + \beta = 0 \\ -DA + \delta B + \varepsilon BA = 0 \\ 2A\delta + \varepsilon A^2 = 0 \end{cases} . \tag{2.8}$$

Using the Wu elimination method yields the following solutions which include two different cases when the parameters satisfy:

$$\beta\varepsilon^3 = 8\alpha\delta^3. \tag{2.9}$$

We find the following two cosine-type compacton solutions for Eq.(1.1):

$$u_1 = \begin{cases} \cos(k(x - Dt)), & |(x - Dt)| \le \frac{\pi}{2k} \\ 0, & |(x - Dt)| > \frac{\pi}{2k} \end{cases} , \tag{2.10}$$

$$v_1 = \begin{cases} -\frac{2\delta}{\varepsilon}\cos(k(x-Dt)) - \frac{\lambda\delta}{\beta\varepsilon}, & |(x-Dt)| \le \frac{\pi}{2k} \\ -\frac{\lambda\delta}{\beta\varepsilon}, & |(x-Dt)| > \frac{\pi}{2k} \end{cases}, \tag{2.11}$$

and

$$u_2 = \begin{cases} \cos^2(\frac{k}{2}(x-Dt)), & |(x-Dt)| \le \frac{\pi}{k} \\ 0, & |(x-Dt)| > \frac{\pi}{k} \end{cases}, \tag{2.12}$$

$$v_2 = \begin{cases} -\frac{2\delta}{\varepsilon}\cos^2(\frac{k}{2}(x-Dt)) - \frac{\lambda\delta}{\beta\varepsilon}, & |(x-Dt)| \le \frac{\pi}{k} \\ -\frac{\lambda\delta}{\beta\varepsilon}, & |(x-Dt)| > \frac{\pi}{k} \end{cases}, \tag{2.13}$$

where $k = \frac{1}{2}\sqrt{\frac{-\lambda^2-4D\beta}{\beta\gamma}}$.

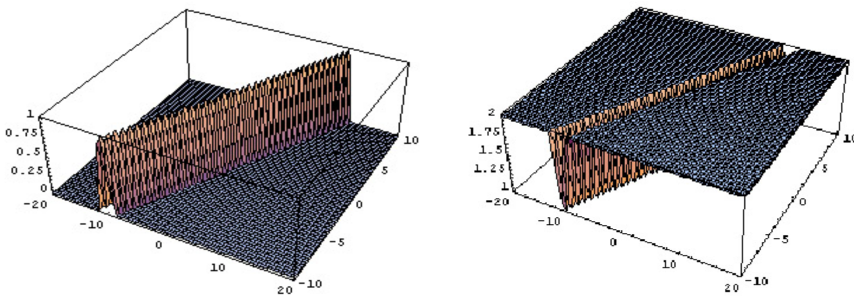

Figure 1: Graphs of solution u_1 and v_1

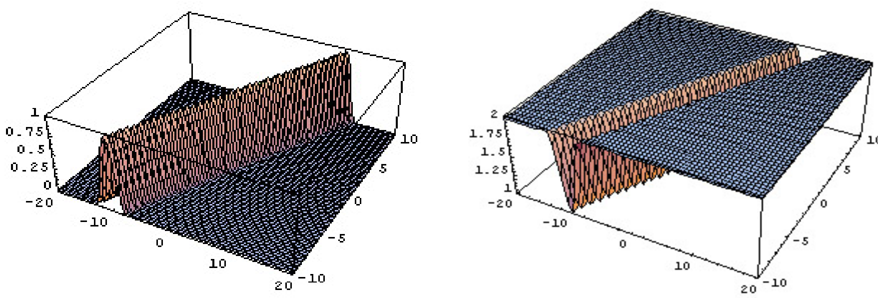

Figure 2: Graphs of solution u_2 and v_2

We can get a view of u_1, v_1 and u_2, v_2 in Fig.1 and Fig.2 with $\varepsilon = 4, \delta = 2, \lambda = 4, \gamma = 2, \beta = -1, d = 1$.

According to formula (2.5), taking the same steps we obtain sin-type compacton solutions for Eq.(1.1)

$$u_3 = \begin{cases} \sin(k(x-Dt)), & |(x-Dt)| \le \frac{\pi}{2k} \\ 1, & (x-Dt) > \frac{\pi}{2k} \\ -1, & (x-Dt) < -\frac{\pi}{2k} \end{cases}, \tag{2.14}$$

$$v_3 = \begin{cases} -\frac{2\delta}{\varepsilon}\sin^2(\frac{k}{2}(x-Dt)) - \frac{\lambda\delta}{\beta\varepsilon}, & |(x-Dt)| \le \frac{\pi}{k} \\ -\frac{2\delta}{\varepsilon} - \frac{\lambda\delta}{\beta\varepsilon}, & (x-Dt) > \frac{\pi}{k} \\ \frac{2\delta}{\varepsilon} - \frac{\lambda\delta}{\beta\varepsilon}, & (x-Dt) < -\frac{\pi}{k} \end{cases}, \tag{2.15}$$

and

$$u_4 = \begin{cases} \sin^2(\frac{k}{2}(x-Dt)), & |(x-Dt)| \le \frac{\pi}{k} \\ 1, & |(x-Dt)| > \frac{\pi}{k} \end{cases}, \tag{2.16}$$

$$v_4 = \begin{cases} -\frac{2\delta}{\varepsilon}\sin^2(\frac{k}{2}(x-Dt)) - \frac{\lambda\delta}{\beta\varepsilon}, & |(x-Dt)| \le \frac{\pi}{k} \\ -\frac{2\delta}{\varepsilon} \;-\frac{\lambda\delta}{\beta\varepsilon},, & |(x-Dt)| > \frac{\pi}{k} \end{cases}, \tag{2.17}$$

where $k = \frac{1}{2}\sqrt{\frac{-\lambda^2-4D\beta}{\beta\gamma}}$.

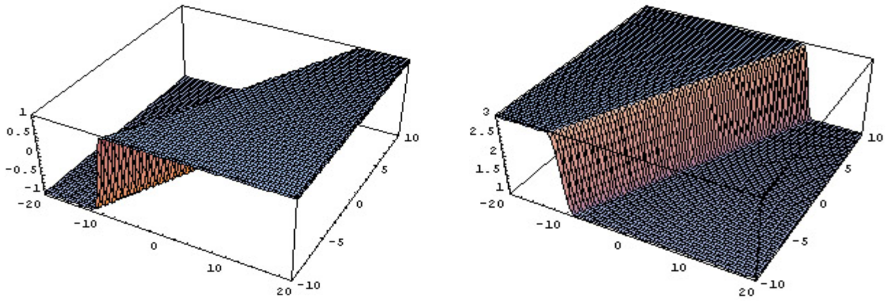

Figure 3: Graphs of solution u_3 and v_3

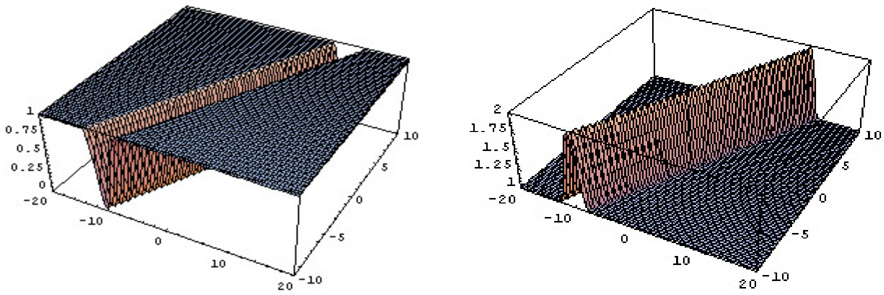

Figure 4: Graphs of solution u_4 and v_4

Given the same parameters $\varepsilon = 4, \delta = 2, \lambda = 4, \gamma = 2, \beta = -1, d = 1$. Fig.3 shows that u_3 and v_3 are kink solutions while Fig.4 shows that u_4 and v_4 are one-peak compacton solutions.

3 Peakon solutions

Integrating (2.3), with respect to the variable ξ, and taking the integrating constant as zero yields

$$\begin{cases} (-D+\alpha B^2A)u + (2A^2\alpha B+\lambda)\frac{1}{2}u^2 + (\alpha A^3+\beta)\frac{1}{3}u^3 + \gamma k^2 u'' = 0 \\ (-DA+\delta B+\varepsilon BA)u + \frac{1}{2}u^2(2\delta A+\varepsilon A^2) = 0 \end{cases}. \tag{3.1}$$

Letting $u = Pe^{-|\xi|}$, where P is constant need to be determined, substituting this into (3.1), we obtain

$$\begin{cases} (-D+\alpha B^2A+\gamma k^2)Pe^{-|\xi|} + \frac{1}{2}(2A^2\alpha B+\lambda)P^2e^{-2|\xi|} + \frac{1}{3}(\alpha A^3+\beta)P^3e^{-3|\xi|} = 0 \\ (-DA+\delta B+\varepsilon BA)e^{-|\xi|} + \frac{1}{2}(2\delta A+\varepsilon A^2)P^2e^{-2|\xi|} = 0 \end{cases}.$$

Setting the coefficients of $e^{-n|\xi|}(n = 1, 2, 3)$ to zero yields a set of algebraic polynomials with respect to unknowns A, B and k,

$$\begin{cases} -D+\alpha B^2A+\gamma k^2 = 0 \\ 2A^2\alpha B+\lambda = 0 \\ \alpha A^3+\beta = 0 \\ -DA+\delta B+\varepsilon BA = 0 \\ 2A\delta+\varepsilon A^2 = 0 \end{cases}. \tag{3.2}$$

Solving Eq.(3.2), we get a peakon solutions of Eq.(1.1) as following demanded

$$\beta\varepsilon^3 = 8\alpha\delta^3, u_5 = e^{-\frac{1}{2}\sqrt{\frac{\lambda^2+4D\beta}{\beta\gamma}}|x-Dt|}, v_5 = -\frac{2\delta}{\varepsilon}e^{-\frac{1}{2}\sqrt{\frac{\lambda^2+4D\beta}{\beta\gamma}}|x-Dt|} - \frac{\lambda\delta}{\beta\varepsilon}.$$

We can get a view of u_5, v_5 in Fig.5 with $\varepsilon = 5, \delta = 1, \lambda = 1, \gamma = 5, \beta = 3, d = 1$.

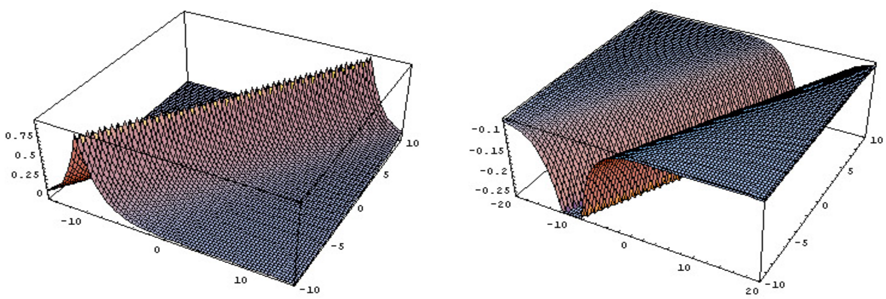

Figure 5: Graphs of solution u_5 and v_5

4 Solitary pattern solutions

We suppose that Eq.(1.1) has the following traveling wave form solutions:

$$u(\xi) = R\cosh^m(\xi), \tag{4.1}$$

and

$$u(\xi) = R\sinh^m(\xi). \tag{4.2}$$

Substituting (4.1) into system (2.3) and collecting all terms with the same power in $\cosh(\xi)$, we get

$$\begin{array}{l} Rm(-D+\alpha B^2 A + k^2m^2\gamma)\cosh^{m-1}(\xi) + (\alpha A^3+\beta)R^3 m\cosh^{3m-1}(\xi) \\ +(2A^2\alpha B+\lambda)R^2 m\cosh^{2m-1}(\xi) - \gamma k^2 Rm(m-1)(m-2)\cosh^{m-3}(\xi) = 0 \end{array}, \tag{4.3}$$

$$(-AD+\delta B+\varepsilon AB)Rm\cosh^{m-1}(\xi) + R^2 m\cosh^{2m-1}(2\delta A+\varepsilon A^2) = 0 \ . \tag{4.4}$$

Letting all coefficients of $\cosh(\xi)$, in Eq.(4.3) and (4.4) to zero we can get a set of algebraic polynomials with respect to the unknown variables $A, B, D, m, R, k, \delta, \alpha, \beta, \varepsilon$.

Using the Wu elimination method yields the following solutions which include two different cases when the parameters satisfy:

$$\beta\varepsilon^3 = 8\alpha\delta^3 \ . \tag{4.5}$$

We find the following solitary-solutions for Eq.(1.1):

$$\begin{cases} u_6 = \cosh(k(x-Dt)) \\ v_6 = \frac{-2\delta}{\varepsilon}\cosh(k(x-Dt)) - \frac{\lambda\delta}{\beta\varepsilon} \end{cases}, \quad \begin{cases} u_7 = \cosh^2(\frac{k}{2}(x-Dt)) \\ v_7 = \frac{-2\delta}{\varepsilon}\cosh^2(\frac{k}{2}(x-Dt)) - \frac{\lambda\delta}{\beta\varepsilon} \end{cases}.$$

According to formula (4.2), taking the same step we obtain

$$\begin{cases} u_8 = \sinh(k(x-Dt)) \\ v_8 = \frac{-2\delta}{\varepsilon}\sinh(k(x-Dt)) - \frac{\lambda\delta}{\beta\varepsilon} \end{cases}, \begin{cases} u_9 = \sinh^2(\frac{k}{2}(x-Dt)) \\ v_9 = \frac{-2\delta}{\varepsilon}\sinh^2(\frac{k}{2}(x-Dt)) - \frac{\lambda\delta}{\beta\varepsilon} \end{cases}.$$

where $k = \frac{1}{2}\sqrt{\frac{\lambda^2+4D\beta}{\beta\gamma}}$.

5 Conclusion

We have considered the coupled nonlinear equations, by applying direct sine method, cosine method, we have succeed in given several kinds of special solitary wave solutions, such as compacton solutions, peakon solutions, as well as traveling solutions.

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